

Quantum Hall transitions: an exact theory based on conformal restriction

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We revisit the problem of the plateau transition in the integer quantum Hall effect. Here we develop a novel analytical approach for this transition, and for other 2D disordered systems, based on the theory of ‘conformal restriction’. This is a mathematical theory that was recently developed within the context of the ‘Schramm-Loewner Evolution’ [SLE] which describes the ‘stochastic geometry’ of fractal curves and other stochastic geometrical fractal objects in two-dimensional space. Observables elucidating the connection with the plateau transition include the so-called *point-contact conductances* [PCCs] between points on the boundary of the sample, described within the language of the Chalker-Coddington network model for the transition. We show that the disorder-averaged PCCs are characterized by a *classical* probability distribution for certain geometric objects in the plane (which we call *pictures*), occurring with positive statistical weights, that satisfy the crucial so-called ‘restriction property’ with respect to changes in the shape of the sample with *absorbing boundaries* - physically, these are boundaries connected to *ideal leads*. At the transition point these geometrical objects (*pictures*) become fractals. Upon combining this ‘restriction property’ with the expected conformal invariance at the transition point, we employ the mathematical theory of ‘conformal restriction measures’ to relate the disorder-averaged PCCs to correlation functions of (Virasoro) primary operators in a conformal field theory (of central charge $c = 0$). We show how this can be used to calculate these functions in a number of geometries with various boundary conditions. Since our results employ only the conformal restriction property, they are equally applicable to a number of other critical disordered electronic systems in two spatial dimension, including for example the spin quantum Hall effect, the thermal metal phase in symmetry class D, and classical diffusion in two dimensions in a perpendicular magnetic field. For most of these systems we also predict exact values of critical exponents related to the spatial behavior of various disorder-averaged PCCs.

I. INTRODUCTION

Effects of static, randomly placed impurities (disorder) are central to our understanding of transport properties of electronic solids. Indeed, building on P. W. Anderson’s seminal work,¹ an immense amount of research activity has emerged over the past few decades on models of disordered electronic solids, in particular of non-interacting electronic systems, the subject now generically known as the Anderson localization². About a decade and a half ago, the field of Anderson localization has received a tremendous boost through the work of Zirnbauer,³ and Altland and Zirnbauer⁴ (AZ) which provided a very general classification scheme of the behavior of non-interacting fermions subject to static disorder potentials. Their work showed that universal behavior emerging on length scales much longer than the mean free path must be in one of only ten possible symmetry classes, which depend solely on the behavior of the Hamiltonian under generic symmetries (time-reversal, particle-hole, chiral)⁵. These ten symmetry classes are in one-to-one correspondence with the ten types of symmetric (constant curvature) Riemannian spaces in the classification scheme of E. Cartan.

Electronic disordered systems exhibit, in a variety of symmetry classes and spatial dimensions, second order quantum phase transitions between insulating and conducting phases, which are examples of Anderson (localization) transitions. Other examples of Anderson (localization) transitions are quantum Hall plateau transitions between insulating phases with different topological order and different quantized values of a Hall conductance. A famous example is the integer quantum Hall (IQH) plateau transition observed in two dimensional semi-conductor devices subject to strong magnetic fields. The nature of the critical state at and the critical phenomena near such an Anderson transition are of great current interest.⁷

In spite of much effort over several decades, an analytical treatment of most of the critical conducting states in disordered electronic systems, including in particular that of the mentioned IQH transition, has been elusive (although some proposals^{9–11} have been put forward – but see Refs. [12,13]). A notable exception is the so-called spin quantum Hall (SQH) plateau transition,^{14,15} which is similar to the IQH transition, but in a different symmetry class (class C in the AZ classification). In this case an exact mapping to the classical prob-

lem of bond percolation is available.¹⁶ Through (variants of) this mapping, exact expressions for various disorder-averaged observables and critical exponents for the SQH transition were obtained.^{16–21}

The universal (critical) properties of Anderson transitions can be formulated in terms of so-called network models. The prime example of a network model is the celebrated Chalker-Coddington network model²² describing IQH transition. The similar network for the SQH transition¹⁴ was the starting point for the mapping to percolation¹⁶ mentioned above. While a network model formulation exists for systems in all ten (AZ) symmetry classes, a particularly rich behavior is seen in class D in two dimensions,^{23–26} comprising, for example, a fermionic representation of the two-dimensional short-range Ising spin glass (Ising exchange couplings with random signs). The phase diagram of a generic network model in class D contains three phases: an insulator, the so-called thermal quantum Hall state, and a metal with continuously varying (thermal) Hall conductivity.

Conventional critical statistical mechanics models are known to possess conformal invariance.²⁷ Implications of this invariance are most powerful in two dimensions, where it allows to apply methods of 2D conformal field theory (in short: CFT)^{28,29} to study critical phenomena in such models. As in any field theory description, basic objects of study in CFT are correlation functions of local observables. One of the more important characteristics in any CFT is the so-called central charge c . This parameter is related to the way a critical system responds to changes in its geometry.

It is widely believed that Anderson transitions in two dimensions also possess conformal symmetry (and there is numerical evidence to support this belief in certain cases.^{31–33}) However, in this case one is usually interested in correlation functions (density of states, conductivities, etc.) averaged over all disorder realizations. Taking such averages is complicated since the partition function of a disordered system undergoes statistical fluctuations from one realization of disorder to another³⁰. A way to handle this difficulty is to apply the supersymmetry method where two types of fields (bosonic and fermionic) are introduced in the theory. The outcome is a theory with the partition function $Z = 1$, independent of the particular disorder realization, and the shape and size of the system. This implies the vanishing of the central charge for a CFT describing an Anderson transition in two dimensions (see e.g. Ref. [34] for a recent review).

Recently, a novel approach to the study of two-dimensional critical systems has appeared. The approach uses methods of probability theory and conformal maps, and can be called ‘stochastic conformal geometry approach’. The focus of stochastic geometry is to directly describe randomly fluctuating geometric objects in scale-invariant (i.e. critical) systems: regions in space of fractal dimension (‘clusters’) and their boundaries (‘cluster boundaries’) which form fractal curves. In a seminal paper³⁵ Oded Schramm has introduced a one-

parameter family of random processes, since then called the Schramm-Loewner evolutions (SLE), which describe growth of random fractal conformally-invariant curves. Conformal invariance in this case is understood precisely as a statement about probability measures on curves. Since the original discovery, the SLE processes have been studied in depth, have been related to the traditional CFT, and generalized in several ways. Many reviews of this beautiful theory exist by now, and we recommend Refs. [36–42] for more details.

The curves described by SLE are unique candidates for (scaling limits of) cluster boundaries in critical statistical mechanics systems in 2D. The one-parameter family SLE_κ discovered by Schramm, where κ is a real parameter, fully exhausts all possible ensembles of SLE curves. The parameter κ of the SLE_κ family is related to the central charge of the critical system by

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}. \quad (1)$$

Since we are interested in theories with $c = 0$, the values $\kappa = 8/3$ and $\kappa = 6$ are special for us. The $\text{SLE}_{8/3}$ describes the scaling limit of self-avoiding random walks (SAW) or polymers, and the SLE_6 describes the percolation hulls. These two types of critical curves possess special properties called locality (for $\kappa = 6$) and restriction (for $\kappa = 8/3$). It is the restriction property that is intimately related to CFTs with $c = 0$. It turns out that the notion of conformal restriction can be extended to certain two-dimensional sets. There is a one-parameter family of conformal restriction measures, supported on these sets, fully characterized by a real number h called the *restriction exponent*. In terms of CFT this exponent is the scaling dimension of a certain boundary primary operator. The sets that are described by conformal restriction all have boundaries (curves) that happen to be versions of $\text{SLE}_{8/3}$, called $\text{SLE}(8/3, \rho)$, where the parameter ρ is related to the exponent h by

$$h(\rho) = \frac{(3\rho + 10)(2 + \rho)}{32}, \quad \rho(h) = \frac{2}{3}\sqrt{24h + 1} - \frac{8}{3}. \quad (2)$$

The theory of conformal restriction and (multiple) $\text{SLE}(\kappa, \rho)$ is presented in Refs. [37,43–50], and we will review relevant results from it later in the paper.

In this paper we propose to use the theory of conformal restriction to study quantum Hall transitions and other 2D disordered electronic systems. A connection between a 2D disordered electron system and the theory of conformal restriction can be established by studying the so-called point contact conductance (PCC), that is the conductance between two infinitely narrow leads introduced and studied in Ref. [51]. Loosely speaking, in a given microscopic model the average PCC is represented as a sum of contributions from paths that the current follows between the point contacts, in the sense of the Feynman path integral (or sum) for a quantum mechanical amplitude. When the contacts are placed at the *boundary* of a

disordered conductor, we obtain the *boundary* PCC. As we will explicitly show, the current paths, when studied in the presence of *absorbing boundaries*, satisfy the restriction property on the lattice (i.e. at the discrete, as opposed to the continuum level). Assuming the discrete (lattice) model has a continuum limit at its critical point, we expect the continuum analogues of the current paths to satisfy the (continuum) restriction property. Furthermore, making the assumption of conformal invariance, we conclude that scaling limits of the current paths can be described by conformal restriction measures. In the following we will keep these assumptions in mind without stressing the difference between discrete and continuous settings.

The fact that the continuum limits of current paths satisfy the restriction property turns out to imply immediately that, in the language of CFT, the point contacts are points of insertions of (Virasoro) *primary* conformal boundary operators. The connection between current paths and restriction measures opens up the possibility to obtain, for the first time, analytical results for average PCC's at the IQH critical point with a variety of boundary conditions. In addition, we show how our results naturally apply to the SQH transition, where the current paths are percolation hulls which, in the continuum limit, are described by SLE and rigorously known to satisfy the restriction property.

Classical diffusion and transport in two-dimensions in a strong perpendicular magnetic field^{53–56} also admit a description in terms of conformal restriction. This is especially clear in the classical limit of the Chalker-Coddington model considered in Ref. [56], where it was shown that in the continuum limit conductances of various kinds can be obtained by solving Laplace's equation with tilted (oblique) boundary conditions. The tilt angle is the Hall angle. This setting is naturally related to Brownian motions reflected at an angle (related to the Hall angle) upon hitting a reflecting boundary. Such reflected Brownian motions underlie microscopic constructions of arbitrary restriction measures.⁴³ A field theory formulation of classical high-field transport was given in the form of a Gaussian model which is the linearized version of Pruisken's (replica) sigma model for the IQH effect.⁵⁶

The same Gaussian model results from linearization of a *different* nonlinear sigma model that describes thermal transport of quasiparticles in disordered superconductors in class D in 2D.^{57–59} The perturbative renormalization group flow in this model is towards weak coupling, and in a finite system of size L one can linearize the model to obtain the Gaussian model with the coupling of order $(\log L)^{-1}$. In this limit quasiparticle transport is essentially classical with thermal conductivities (divided by temperature, and in the corresponding units) growing logarithmically with length scale, $\sigma_{xx} \sim \log L$, while σ_{xy} is arbitrary. Thus, our results obtained from the general theory of conformal restriction apply to this system as well.

Before we proceed with a detailed derivation of our results, we briefly summarize them here. The main results that apply to all systems that we have mentioned above are:

1. Average PCCs within microscopic models are mapped to *classical* statistical mechanics problems with *positive, albeit in some cases nonlocal, weights*.
2. The weights so obtained are *intrinsic* and satisfy the crucial *restriction property* with respect to deformations of *absorbing* boundaries, details to be explained below.
3. Current insertions through point contacts on a boundary are (Virasoro) *primary* CFT operators. The dimensions of these operators are known exactly in some cases, and numerically in others.
4. Assuming conformal invariance we find exact functional forms of average PCCs in various geometries.

The rest of the paper is organized as follows.

In Section II we explain the conformal restriction property. We also explain in general terms how the graphical representation of boundary PCCs in terms of Feynman paths satisfies restriction with respect to absorbing boundaries.

For each of the models mentioned above (IQH, SQH, diffusion in strong magnetic field, and the metal in class D), we provide a detailed derivation of the relation between the averaged PCCs and classical weights satisfying the restriction property in Section III. Specifically, in Section III A we will explicitly show how the construction outlined in Section II works for the disorder-averaged PCCs in the Chalker-Coddington model for the integer quantum Hall plateau transition.²² We do the same for the network model for the SQH transition¹⁴ through the mapping to classical percolation¹⁶ in Section III B, then for the classical limit of the CC model in Section III C, and for the metal in class D in Section III D.

The next Section IV is devoted to a presentation of the theory of conformal restriction, multiple SLEs. Section V sets up some useful notation and explains the relation of conformal restriction and SLEs to CFT in the so-called Coulomb gas formalism.

In the following Section VI we make use of the conformal restriction theory to obtain certain information on the transport behavior of the systems of interest. We establish the functional forms of average PCCs in several geometries. We also compute the conformal weights (scaling dimensions) of some of the relevant primary operators. Some of these weights turn out to be *super-universal* in the sense that they are fully determined by conformal restriction alone, and do not depend on the particular symmetry class of the model (see TABLE I for a summary).

In Section VII we discuss our results with the view on possible extensions and generalizations. Two appendices provide some relevant background information from graph theory, and details of some calculations.

II. CONFORMAL RESTRICTION AND MODELS WITH $c = 0$

We begin this section by describing the conformal restriction property. Then we explain how current paths contributing to boundary PCCs at Anderson critical points naturally satisfy this property with respect to *absorbing* boundaries (which, we recall, describe ideal leads attached to the boundaries).

A. Conformal restriction property

Consider a statistical ensemble of curves defined in a simply-connected domain D of the complex plane. All these curves start at a fixed point a on the boundary of D and end at another fixed boundary point b , see Fig. 1. The ensemble is specified by a finite measure on the curves. The measure can be normalized to be a probability measure, but it is more natural and convenient to think about un-normalized weights associated with curves in the ensemble, similar to Boltzmann weights of configurations in statistical mechanics.

Next, consider a set A such that the topology of the sub-domain $D \setminus A$ is the same as that of D . This means that A is “attached” to the boundary of D , so that $D \setminus A$ is simply-connected, and that the points a and b belong to the parts of the boundary that are common between D and $D \setminus A$. Notice that we allow for sets A that have more than one connected component.

The original ensemble of random curves can be used to define two new ensembles of curves in the sub-domain $D \setminus A$. The first one is obtained by restriction: it is the ensemble of curves in D conditioned not to intersect A . In other words, of all the curves in the original ensemble we keep only those that do not enter A . To a curve $\gamma \in D \setminus A$ this definition assigns the same weight in the new ensemble that this curve has in the original ensemble. The second way to define a new ensemble in the sub-domain is to choose a conformal map Φ from $D \setminus A$ to D that fixes the points a and b ($\Phi(a) = a$, $\Phi(b) = b$), and to any curve $\gamma \in D \setminus A$ assign the weight of its image $\Phi(\gamma)$ in the original ensemble. This is called the conformal transport of the probability measure.

Now the original ensemble is said to satisfy the conformal restriction property if both ways of defining a new ensemble in the sub-domain $D \setminus A$ lead to the same result for *any* set A of the type described above. If in this construction we use sets A that can only border the boundary of D on one arc from a to b , say, the one that goes counterclockwise, then we have the so-called one-sided restriction, see the left panel in Fig. 1. If different connected components of A can be attached to either of the arcs of the boundary, we have the two-sided restriction, see the right panel in Fig. 1. Notice that this is a stronger property since any two-sided restriction measure automatically satisfies the one-sided restriction, but the opposite is not necessarily true.

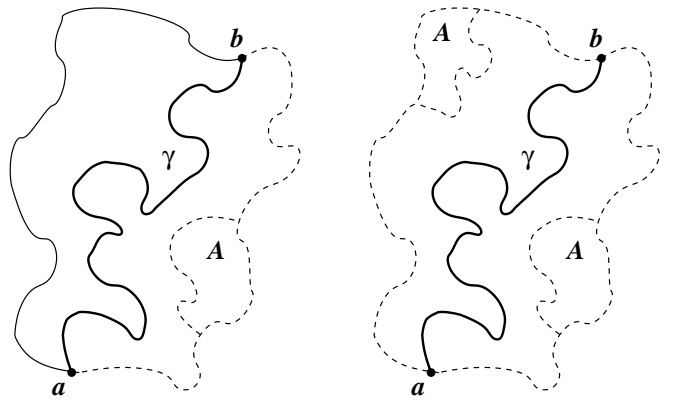


FIG. 1: Restriction property for curves. A curve γ in D is restricted not to enter the sets A . The portions of the boundary where the set A can be attached are shown by dashed lines. They correspond to *absorbing* boundaries in the physical models we consider. Left: one-sided restriction. Right: two-sided restriction.

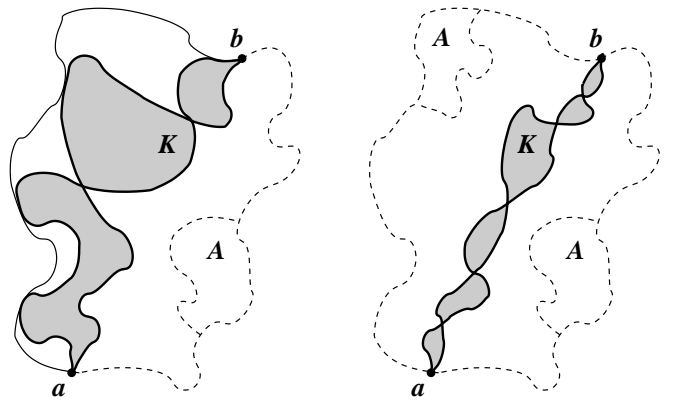


FIG. 2: Restriction property for sets. A compact set K in D is restricted not to enter the set A . The portions of the boundary where the sets A can be attached are shown by dashed lines. They correspond to *absorbing* boundaries in the physical models we consider. Left: one-sided restriction. Right: two-sided restriction.

It is known⁴³ that the only ensemble of simple curves that satisfies the two-sided conformal restriction property is the $\text{SLE}_{8/3}$. However, we can consider more general sets $K \subset D$ that “touch” the boundary of D only at the two fixed points a and b , see the left panel in Fig. 2. In this case we get a one-parameter family of two-sided restriction measures (that is, statistical ensembles of such sets, or clusters, K) characterized by the restriction exponent h . If the sets K are allowed to “touch” the boundary only along, say, the clockwise arc from a to b , then we get more general one-sided restriction measures, see the right panel in Fig. 2. All restriction measures are fully classified, and, moreover, there is an explicit construction of all of them.⁴³

The restriction property defined by the condition of avoidance of sets A attached to the boundary immedi-

ately implies the following. Consider the right panel in Fig. 2. It is clear that a set K intersects A if and only if its boundary (shown by thick curves) intersects A . Thus, the restriction property does not care about the internal structure of the set K , and it is sufficient to “fill it in” and consider the boundaries of the filled-in sets. This means that two different ensembles of sets that only differ by their internal structure but have the same fillings and boundaries lead to the same restriction measure. An example of this is provided by ensembles of Brownian excursions and percolation hulls conditioned to avoid the boundary (see Sec. III below for details).

We note in passing that for a certain range of the restriction exponent h ($h < 35/24$) samples of two-sided conformal restriction measures (the filled-in sets K) have so-called *cut points*.⁴⁴ These are points with the property that if one of the them is removed, the filled-in set K becomes disconnected. These points are shown on the right panel in Fig. 2 as intersections of the “left” and “right” boundaries of K . These points are similar to the so-called “cutting bonds”,⁶⁰ which are important components in the structure of percolation clusters. In fact, in the mapping to percolation for the SQH transition, the cut points are exactly the cutting bonds of the critical percolation clusters.

For a one-sided restriction measure (see the left panel in Fig. 2), its sample may “touch” the portion of the boundary where we are not attaching sets A . Then all statistical information related to the restriction property is encoded in the “left” filling of the set K or, equivalently, in its “right” boundary. In either case, as we have mentioned in the introduction, the boundaries of restriction measures are variants of $\text{SLE}_{8/3}$ known as $\text{SLE}(8/3, \rho)$. We shall give more details on $\text{SLE}(\kappa, \rho)$ in Section IV.

As we have mentioned, it is more natural to think of un-normalized restriction measures. In this case the total weight of a restriction measure can be thought of as a partition function $Z_D(a, b)$ that is the sum of weights of all sets K in the ensemble. This partition function depends on the domain D , the marked points a and b on the boundary ∂D where the random set K intersect the boundary, and on the type of the restriction measure that we consider.

B. Critical curves at $c = 0$ and disordered systems

Consider now a finite 2D disordered conductor occupying a domain D . We can place small contacts at points a and b on the boundary of D and measure the boundary PCC $g(a, b)$ between the contacts. In this paper we will focus specifically on systems described by network models of Chalker-Coddington (CC) type (see Fig. 3). Then, in general, a diagrammatic approach can be developed for computing the average conductance $\langle g(a, b) \rangle$ (more details will be presented below for specific models). In particular, “Feynman” paths drawn on a network for a

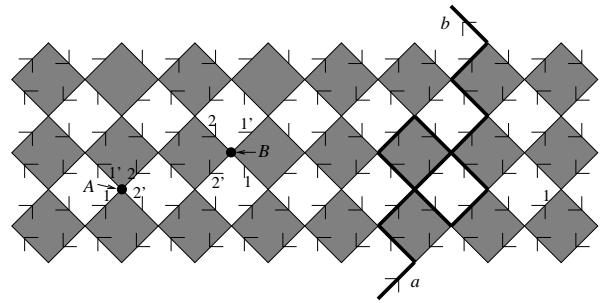


FIG. 3: The Chalker-Coddington network model. The fluxes propagate on the links in the directions shown by the arrows. The bold line connecting the links a and b on the boundary represents a Feynman path or a picture (see main text) contributing to the point contact conductance $g(a, b)$ between these two contacts.

system defined in the domain D determine contributions to $\langle g(a, b) \rangle$. All these paths begin at the point a and end at the point b and are connected, which is a crucial feature of a disordered system. Indeed, for a system with quenched disorder we must average not the partition function, but the free energy. While the partition function generates all paths, the free energy generates connected paths only.

Let us examine under what conditions the Feynman paths satisfy restriction. In order to do so we consider two sets of Feynman paths:

1. The Feynman paths for $\langle g(a, b) \rangle$ for the system defined in the domain $D \setminus A$.
2. The Feynman paths for $\langle g(a, b) \rangle$ in D which do not enter A .

It is easily seen that paths from the two sets will have the same weight after disorder averaging if the rules for generating the paths do not depend explicitly on the domain in which they are defined. The only dependence on the domain is that the paths are drawn in that domain. In other words, the crucial condition for a set of curves to satisfy restriction is that the weights of the curves are *intrinsic*.

An important caveat has to be added to this statement: special boundary conditions must be chosen in order to allow us to identify the weight of paths described in items 1 and 2 above. These boundary conditions may be described as the “absorbing boundary conditions” and often come up naturally in the study of disordered systems: they describe, as already mentioned, the presence of ideal leads attached to the boundary. Indeed, a given path may approach the boundary, and then a certain weight will be associated with the path turning back into the bulk or escaping the system through the boundary. For network models, these weights are determined by parameters ascribed to a particular node that is on the boundary of the network. For the weights to be intrinsic, they must not

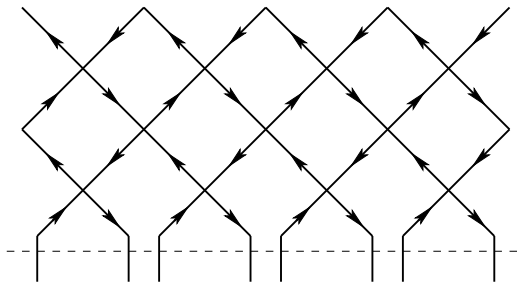


FIG. 4: CC network with absorbing boundary.

depend on whether a particular node lies on the boundary or in the bulk of the system. Therefore, a boundary node should be such as to allow a path going through that node to escape the system. A boundary with such nodes is called *absorbing*, and in physical terms it is realized by attaching ideal leads to the disordered system. A microscopic picture of the absorbing boundary for the CC network is shown in Fig. 4. The role and importance of the absorbing boundary conditions will be described in more detail in Section III, where we shall also describe in more detail the assignment of classical statistical weights to sets of Feynman paths, after disorder averaging.

As in the formal definition of the restriction property above, it is sufficient to consider not the Feynman paths themselves with all their internal structure (multiple loops and crossings), but their fillings and boundaries. This will be implicitly assumed in the following. In particular, in the case of the CC model considered in Sec. III A we will introduce the “pictures” that emerge as important geometric objects determining contributions to conductances. They will derive from disorder-averaged pairs of Feynman paths on the links of the network model, and will have loops. However, as for any sets satisfying restriction, it will be sufficient to consider the filled pictures and their boundaries, especially in the continuum limit. In the presentation below we sometimes will refer to both filled and unfilled objects as “pictures” to simplify the discussion. However, when we need to distinguish a picture and its filling, we will make the distinction explicit.

In order to establish conformal restriction we must assume that an alternative way to obtain the paths in the first set (in item 1 above) is to conformally map the paths from domain D onto domain $D \setminus A$. But this assumption is the standard assumption of conformal invariance of critical systems in two dimensions. So we expect conformal restriction to hold for the set of averaged Feynman paths for a disordered system at criticality. In order to connect this conformal restriction property to probability theory we must also show that the weights obtained for configurations of paths after disorder averaging are positive, such that they can be considered as *classical* statistical weights. This will hold in the systems of interest to us, as discussed below in Section III. We expect that a similar formulation, utilizing Feynman paths, is possible for a

large class of network models describing other disordered systems.

In all cases that we consider the relevant classical geometric objects describing average PCCs at critical points become samples of conformal restriction measures in the continuum limit. This immediately leads to the following consequences. First, this means that the average PCCs are equal (up to some normalization factor) to the partition functions that we have introduced above:

$$\langle g(a, b) \rangle = Z_D(a, b). \quad (3)$$

This relation alone has very strong implications. We will see in Section IV that the partition functions Z_D transform as two-point functions of (Virasoro) primary operators, at positions a and b , under conformal maps. This turns out to imply that the current insertions at the absorbing boundary (for a two-sided restriction) or at a juxtaposition of the absorbing and a reflecting boundary (for a one-sided restriction) are primary CFT operators, and one can use CFT tools to study their correlation functions.

Secondly, the boundaries of the relevant classical objects are described by $\text{SLE}(8/3, \rho)$. Based on the specific physical situations that we consider, the parameter ρ in this description can take three possible values that we will call ρ_A , ρ_{RA} , and ρ_{LA} . The first of these, ρ_A , corresponds to the two-sided restriction measure, which is relevant for a point contact placed at the absorbing boundary. The other two values correspond to one-sided restriction measures that appear when we place a point contact at a juxtaposition of the absorbing boundary with two possible reflecting boundaries. These two possible reflecting boundaries appear due to the fact that we consider network models with a directionality (an ‘arrow’) on the links [designed to capture the physics of conductors with broken time reversal invariance]. Thus, we can have “right” and “left” reflecting boundaries that would, away from the critical point, support “edge states” propagating towards the point contact or away from it, correspondingly (see Fig. 5). More precisely, to distinguish the two types of reflecting boundaries, we introduce the following notation. Let \hat{n} be the inward normal unit vector, \hat{z} be the unit vector along the z axis normal to the plane of the network, and $\hat{t} = \hat{n} \times \hat{z}$ a unit vector tangential to the boundary. The triple $\hat{t}, \hat{n}, \hat{z}$ is a right-hand triad. The vector \hat{t} can be in the direction of the current flow along the boundary, or can be opposite to it, and this is the distinction between the two types of reflecting boundaries. We will call a reflecting boundary “right” if the direction of the current flow at the boundary is along \hat{t} . Similarly, on a “left” boundary the current flows opposite to \hat{t} . If we introduce x and y coordinates in Fig. 5 is the usual way, then \hat{t}, \hat{n} will be the unit vectors in the x and y direction, respectively. The reflecting lower boundaries in the top (bottom) panel of Fig. 5 are examples of “right” (“left”) boundaries.

We will argue in Section VI that in all models consid-

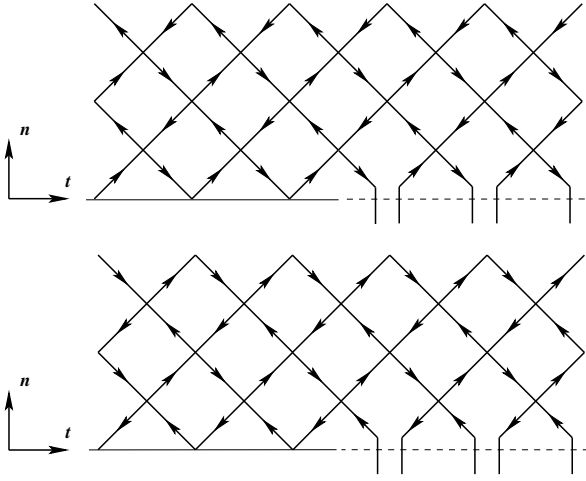


FIG. 5: CC networks with two types of boundary condition changes. Top (bottom): “right” (“left”) boundary juxtaposed with the absorbing boundary.

ered in this paper, the value $\rho_A = 2/3$, which turns out to give a scaling dimension $h_A = 1$ for the conserved current operator. At the same time, the values of ρ_{RA} and ρ_{LA} (h_{RA} and h_{LA}) are known analytically only for the SQH critical point, as well as for the classical limit of the CC model. For the IQH transition these values are known from numerical simulations of the CC model.⁶¹

III. RESTRICTION IN SPECIFIC MODELS OF DISORDERED ELECTRONIC SYSTEMS

In this section we present a detailed analysis of boundary PCC’s in four models of disordered electronic systems: the Chalker-Coddington (CC) model for the IQH transition, the $SU(2)$ network model for the SQH transition, the classical limit of the CC model for diffusion in strong magnetic fields, and a weakly coupled sigma model for a metal in class D.

A. Feynman paths and restriction in the Chalker-Coddington model

In Ref. [22] Chalker and Coddington proposed the following network model to describe the IQH plateau transition. The network consists of links and nodes as shown in Fig. 3. The links carry complex fluxes z_i , and the nodes represent (unitary) scattering matrices \mathcal{S} connecting incoming (z_1, z_2) and outgoing ($z_{1'}, z_{2'}$) fluxes:

$$\begin{pmatrix} z_{1'} \\ z_{2'} \end{pmatrix} = \mathcal{S} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (4)$$

For the time being the scattering amplitudes α, \dots, δ are assumed to be complex numbers constrained only by the unitarity of \mathcal{S} , different for different nodes, which

allows us to formulate our model for disordered samples with any realization of disorder. A particular distribution for the scattering amplitudes will be specified later.

There are two types of nodes in the network, the A - and the B -nodes, which live on one (A) or the other (B) sublattices of nodes, as indicated on Fig. 3. Unitary scattering matrices always admit the so called polar decomposition which for the sublattice S (A or B) is written as follows:

$$\mathcal{S}_S = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \sqrt{1-t_S^2} & t_S \\ -t_S & \sqrt{1-t_S^2} \end{pmatrix} \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix}. \quad (5)$$

Such parametrization is redundant, but when the scattering matrices are multiplied together, the elements of the diagonal unitary matrices are combined in such a way that the resulting phase factors are associated with links rather than with nodes. In the CC model these link phases are assumed to be independent random numbers uniformly distributed between 0 and 2π . Note also that the only negative entry in the nodal scattering matrix (the middle factor in Eq. (5)) corresponds to the scattering from the lower incoming channel (labeled 1) to the lower outgoing channel (labeled 2') in the usual pictorial representation of the CC network (see Fig. 3).

Parameters t_A, t_B have a simple probabilistic meaning: t_A^2 is the probability to turn right upon reaching an A node, and t_B^2 is the probability to turn left upon reaching a B node. The model is isotropic when the possible values of the nodal parameters t_S are related by

$$t_A^2 + t_B^2 = 1. \quad (6)$$

When this equality is satisfied, the probabilities for turning left (or right) at a node are the same for the two sublattices of nodes. Then, depending on whether $t_A < t_B$, at large scales the system flows to either insulating state with zero two-probe conductance g , when all the states are localized, or to the quantum Hall state, when only the bulk states are localized, but there are edge states giving a quantized value of g . The transition between these regimes happens (by symmetry) when $t_A = t_B = 2^{-1/2}$. This determines the critical point in the isotropic CC model.

Let us label the links of the network by integers j . We define an (open) Feynman path f to be an ordered sequence $j_1 = a, j_2, \dots, j_{N(f)+1} = b$ of oriented links on the network that form a continuous path from link a to link b , where a and b are distinct. Here $N(f) + 1$ is the total number of links in the path f . Then $N(f)$ is the number of turns along the path f , which is the same as the number of times the path goes through a node. A given link j can be traversed multiple times $n_j(f)$ in a given path f , except for the first and the last links, where $n_a = n_b = 1$. In other words, $j_k \neq a, b$ for $k = 2, 3, \dots, N(f)$.

For the CC model away from the critical point we need to separately keep track of the number of left (L) and right (R) turns on each sublattice of nodes. Denoting

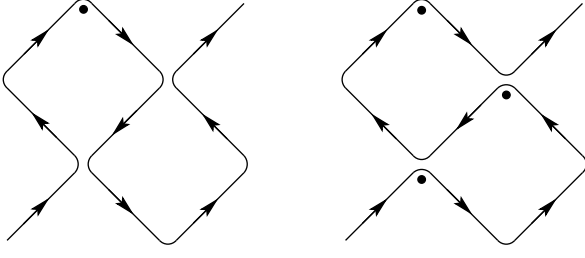


FIG. 6: Two paths that correspond to the same picture. Dots mark the turns that contribute with the minus sign, so that for the path on the left $N_-(f) = 1$, and for the one on the right $N_-(f) = 3$.

these numbers for a given path f by $N_{A,R}(f)$, $N_{A,L}(f)$, $N_{B,R}(f)$, and $N_{B,L}(f)$, we obviously have $N_{A,R}(f) + N_{A,L}(f) + N_{B,R}(f) + N_{B,L}(f) = N(f)$.

If we “forget” the order of the links traversed by a path f , but retain the multiplicity $n_j(f)$ of each link, we get what we will call a “picture”. More generally, a picture p is a map from a subset of all links to a set of positive integers $p : j \rightarrow n_j$. In other words, a picture p can be represented by positive integers n_j associated with some links on the network. It is actually more convenient to associate $n_j = 0$ with the links that do not belong to a picture. This convention will allow us to write unrestricted summations over the links of the network.

It is clear that every path f gives rise to a picture $p(f)$. However, two or more paths that traverse the same set of links in different order, will correspond to the same picture. The simplest example is given by the figure “eight” shown in Fig. 6. Moreover, there are pictures that do not come from any legitimate path. For example, if the sum of the integers n_j on the two incoming links is not equal to the sum of the integers on the two outgoing links at a given node, the picture with such integers cannot come from a legitimate path. We denote by $F(p)$ the set of all paths that give rise to a given picture p . Thus, for all $f \in F(p)$ we have $p(f) = p$, and for some pictures p the set $F(p)$ is empty. In fact, there is a precise relation between pictures and Feynman paths outlined in the Appendix A, where, in particular, we show how many distinct paths correspond to a given picture.

In what follows we will only encounter pictures that come from Feynman paths. Note that for all $f \in F(p)$ the number of turns $N(f)$ is the same:

$$N(f) = \sum_{j \in f} n_j - 1 = N(p), \quad f \in F(p). \quad (7)$$

Thus, this number (as well as the individual link numbers $n_j(f) = n_j(p)$) characterizes a picture rather than a single path f , and we will emphasize this by denoting this number by $N(p)$ whenever appropriate.

Now we consider the quantum mechanical amplitude $A_f(a, b)$ for a path f that goes from a link a to a link b on the network. The amplitude is given by the product

of the phase factors $e^{in_j \phi_j}$ for each link $j \in f$, and matrix elements of the scattering matrices of the nodes encountered by the path f . Let $N_-(f)$ be the number of turns that contribute a negative factor $-t_S$ to the amplitude. Then we have

$$A_f(a, b) = (-1)^{N_-(f)} t_A^{N_{A,R}(f)} (1 - t_A^2)^{N_{A,L}(f)/2} \times t_B^{N_{B,L}(f)} (1 - t_B^2)^{N_{B,R}(f)/2} e^{i \sum_j n_j(f) \phi_j}. \quad (8)$$

The total amplitude $A(a, b)$ for getting from link a to link b is given by the sum of $A_f(a, b)$ over all the paths that go from a to b (we denote this set by \mathcal{F}_{ab}):

$$A(a, b) = \sum_{f \in \mathcal{F}_{ab}} A_f(a, b). \quad (9)$$

We can rewrite this sum by breaking it into the sum over all pictures that come from any of the paths in \mathcal{F}_{ab} [we denote this set of pictures by $P(\mathcal{F}_{ab})$], and the subsequent sum over all the paths giving rise to a specific picture:

$$A(a, b) = \sum_{p \in P(\mathcal{F}_{ab})} e^{i \sum_j n_j(p) \phi_j} S(p), \quad (10)$$

where

$$S(p) = \sum_{f \in F(p)} (-1)^{N_-(f)} t_A^{N_{A,R}(f)} (1 - t_A^2)^{N_{A,L}(f)/2} \times t_B^{N_{B,L}(f)} (1 - t_B^2)^{N_{B,R}(f)/2}. \quad (11)$$

For the isotropic model (for which $t_A = t_B$) this simplifies to

$$S(p) = \sum_{f \in F(p)} (-1)^{N_-(f)} t_A^{N_R(f)} (1 - t_A^2)^{N_L(f)/2}, \quad (12)$$

where $N_R(f)$ and $N_L(f)$ are the total numbers of right and left turns along the path f . At the critical point of the isotropic model this further simplifies to

$$S(p) = 2^{-N(p)/2} \sum_{f \in F(p)} (-1)^{N_-(f)}. \quad (13)$$

A physically relevant observable is the point contact conductance (PCC) $g(a, b)$ between the links a and b given by $g(a, b) = |A(a, b)|^2$. It is worth pointing out that the amplitude $A(a, b)$ determining the PCC is different from the Green’s function $G(a, b)$ (the propagator) between the two points. The PCC in the CC model is defined⁵¹ by cutting the two links a and b of the network and using the resulting open half-links as sources and drains for the current. Thus, the PCC, as any other conductance, is a property of an open system, while the Green’s function is a property of a closed system. The difference is also manifest in the graphical representation of the two quantities: while the PCC gets contributions only from open Feynman paths that go through the initial and the final links only once, the Green’s function

would include all paths between the links.

The conductance $g(a, b)$ is a random quantity that depends on all the phases ϕ_j . In the following we will only be concerned with the disorder-averages of PCC's over the distribution of the phases. We will denote such averages by angular brackets. Using the representation (10) of the propagator as a sum over pictures, we can write

$$g(a, b) = \sum_{p_1, p_2 \in P(\mathcal{F}_{ab})} e^{i \sum_j [n_j(p_1) - n_j(p_2)] \phi_j} S(p_1) S(p_2). \quad (14)$$

Averaging⁵² this expression over the random phases ϕ_j forces the numbers on each link to be the same for the pictures p_1 and p_2 . This can be written as

$$\left\langle e^{i \sum_j [n_j(p_1) - n_j(p_2)] \phi_j} \right\rangle = \prod_j \delta_{n_j(p_1), n_j(p_2)} = \delta_{p_1, p_2}, \quad (15)$$

which implies that different pictures do not interfere when we compute their contributions to $\langle g(a, b) \rangle$. Therefore, we obtain the following expression for the PCC

$$\langle g(a, b) \rangle = \sum_{p \in P(\mathcal{F}_{ab})} S^2(p) = \sum_{p \in P(\mathcal{F}_{ab})} W(p). \quad (16)$$

The significance of this formula is that the disorder-averaged PCC $\langle g(a, b) \rangle$ is represented as a sum of positive quantities $W(p) = S^2(p)$ which can be interpreted as classical positive probability weights associated with pictures p .

We note here in passing that the quantity $S(p)$ is the sum of the amplitudes for the paths $f \in F(p)$ in the CC model where all the link phases are set to zero. It is known that this (non-random) model without phases on the links belongs to class D in the AZ classification, and is equivalent to the non-random 2D (doubled) Ising model,⁶² equivalent to free Dirac fermions. Therefore, the sum $A_0(a, b) = \sum_{p \in P(\mathcal{F}_{ab})} S(p)$ can possibly be computed explicitly by diagonalizing the transfer matrix for this non-random network model. The sum is real, and its square $A_0^2(a, b) = g_0(a, b)$ (the point contact conductance of the CC model without link phases) is straightforward to compute

$$g_0(a, b) = \left(\sum_p S(p) \right)^2 = \sum_p S^2(p) + \sum_{p_1 \neq p_2} S(p_1) S(p_2), \quad (17)$$

which differs from the average conductance of the actual CC model, Eq. (16), where the second summand is absent.

The pictures arising from Feynman paths of the Chalker-Coddington model, as defined above, can be seen to satisfy restriction. Consider the average $\langle g(a, b) \rangle$, where a and b are on the boundary of the sample; the sample is defined in the domain D as in Fig. 2. As-

sume that the boundary conditions are absorbing on that part of the boundary which goes counter-clockwise from a to b (see the left panel in Fig. 2 and Fig. 4). The choice of absorbing boundary conditions is crucial. Electrons approaching a node on the absorbing boundary can continue their path to the outside of the boundary thus “leaking out”. As a consequence, the scattering matrix on the boundary remains to be given by the middle factor in (5), just as in the bulk. The fact that the scattering matrix is the same both at the boundary and in the bulk, and the fact that the statistical weights are intrinsic to the pictures, together ensure the restriction property. Indeed, let A be a set appropriate for the definition of one-sided restriction, as depicted in the left panel in Fig. 2. Then the pictures contributing to $\langle g(a, b) \rangle$ for the system occupying the domain $D \setminus A$, are the same as the pictures contributing to $\langle g(a, b) \rangle$ for the system occupying the domain D which do not enter A , with the same weights.

We note that the restriction property is satisfied in the CC model at the discrete level [i.e. at the level of the (CC) lattice model], and even away from criticality. At the critical point in the continuum we can, in addition, assume conformal invariance. Consequently, we obtain one-sided conformal restriction for (the continuum limit of the) pictures. This property immediately implies the following important result: the contacts where we inject and extract currents in the CFT description in the continuum become insertions of (Virasoro) *primary* operators with certain dimensions h_{LA} or h_{RA} (depending on the type or the reflecting boundary) next to the contact. This also means that the right boundary of a picture is described by SLE(8/3, ρ_{LA}) or SLE(8/3, ρ_{RA}) for some values of ρ_{LA}, ρ_{RA} related to h_{LA}, h_{RA} by Eq. (2).

Note that in order to establish one-sided conformal restriction, little had to be known about the actual weights of the Feynman paths, or the quantum nature of this problem. However a few properties were essential:

- (i) the weights of pictures are intrinsic, they do not depend on the shape of the domain and, are determined by the same rules on the boundary and in the bulk of the system,
- (ii) the weight of each picture is a positive quantity,
- (iii) the pictures are connected: loops, or “vacuum to vacuum” diagrams are absent. This is related to the vanishing of the central charge.

We will see shortly that the same properties hold for paths in other network models.

The above arguments also hold when the whole boundary of the system is absorbing. In this case the pictures are seen to satisfy two-sided restriction, and in the continuum they are created by insertions of certain primary boundary operators of dimension h_A . Their boundaries (both left and right) are then described by SLE(8/3, ρ_A). We will argue in Section VI that $\rho_A = 2/3$ and $h_A = 1$.

As we have mentioned above, the parameter ρ for one-sided restriction can take two possible values ρ_{RA} and ρ_{LA} , depending on the two types of reflecting boundary conditions on that part of the boundary which goes clockwise from a to b (see Fig. 5 above). In the CC model we cannot at present analytically determine the values of ρ_{RA} and ρ_{LA} from microscopic considerations. They can be found numerically,⁶¹ and then various critical exponents and correlation functions will be determined by these values and the theory of conformal restriction. At the same time, in the other three models described below, these values are known exactly, and the theory that we present for those models is complete.

We have already mentioned that the restriction property can be completely formulated in terms of the boundaries of filled-in pictures. Therefore, one can further rearrange the sum over pictures in Eq. (16) by grouping together pictures which have the same fillings. Labeling such fillings by K (similar to the notation for samples of restriction measures used above - see Fig. (2)) and denoting the set of pictures with the same filling K as $\mathcal{P}(K)$, we can write

$$\langle g(a, b) \rangle = \sum_K W(K), \quad W(K) = \sum_{p \in \mathcal{P}(K)} W(p). \quad (18)$$

Notice that while the sum over pictures in Eq. (16) is infinite even for a finite network, the corresponding sum over fillings is in this case finite.

Let us mention here that at the microscopic level the cut points discussed in Section II A correspond to particular links in a filling K . Namely, these are links j that have multiplicity one (i.e. $n_j = 1$) in *each* picture $p \in \mathcal{P}(K)$. It is easy to see that a filling can be broken into “irreducible” parts by removing the “cut links”, and that the probability weight of a filling is given by the product of the weights of its irreducible components.

We finish this section with a comment about a recent paper by Ikhlef, Fendley, and Cardy, Ref. [63]. In this paper the authors consider a certain truncation of the CC model and its integrable deformations. We notice here that this truncation has a natural interpretation in our language of pictures. Namely, it is equivalent to keeping only those pictures where all link multiplicities $n_j(p) \leq 1$. In this case it is actually easy to show that the sign factors $(-1)^{N_-(f)}$ are the same for all the paths $f \in F(p)$ contributing to a given picture p . (The numbers $N_-(f)$ do depend on a particular path, but their parity is the same for all the paths $f \in F(p)$.) However, the truncated model with $t_A = t_B = 1/\sqrt{2}$ happens to be non-critical. To make it critical, one has to introduce an additional weight z for every visited link, and tune z to a particular value z_c . As a consequence, at the critical point of the isotropic truncated model the weight of a picture is given by

$$W(p) = S^2(p) = \left(\frac{z_c}{2}\right)^{N(p)} |F(p)|^2, \quad (19)$$

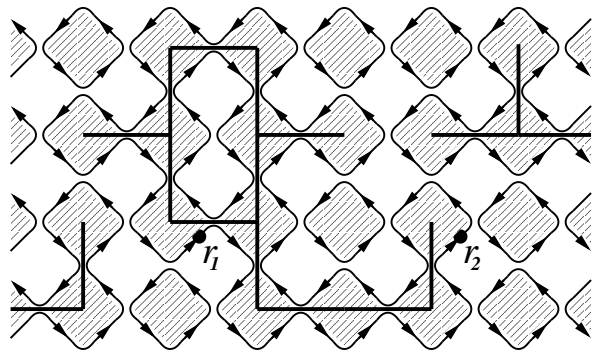


FIG. 7: The average point contact conductance between the links at r_1 and r_2 is given by the probability that these points are connected by a percolation hull.

where in this case $N(p) + 1$ is simply the number of links in the picture, and $|F(p)|$ is given by the first factor in Eq. (A4) from Appendix A, $|F(p)| = \det L_k$. (In the second factor the outdegrees of all the vertices are either 1 or 2, and all the edge numbers $m_{ij} = 1$.) One can also consider “higher” truncations, where only the pictures with $n_j(p) \leq k$ are kept. It is easy to see that all such truncated models, including the one in Ref. [63], satisfy the restriction property on the lattice, and, therefore, should be described by conformal restriction theory (with exponents depending on the truncation level k) at their conformally-invariant critical points (possibly obtained again by some fine-tuning of the link weights).

B. Network model for the spin quantum Hall effect

The spin quantum Hall (SQH) transition was first studied, numerically, in Ref. [14], and a simple physical picture of the SQH effect was given in Ref. [15]. Average conductances within the network model employed in Ref. [14] can be obtained exactly through a mapping to bond percolation on a square lattice.^{16–21} An example of a percolation configuration that appears in the mapping is shown in Fig. 7. Within this approach, average point contact conductances that we focus on in this paper, are explicitly given in terms of probabilities in the percolation problem. These probabilities are intrinsic probabilities of percolation hulls that join the two point contacts. As such, they satisfy restriction property with respect to absorbing boundaries, similar to the weights of the pictures in our treatment of the CC model above.

Specifically, consider the average PCC $\langle g(a, b) \rangle$ between the links a and b , at the critical point of the SQH network model. As is shown in Ref. [20], this is equal to

$$\langle g(a, b) \rangle \propto P(a, b) = \sum_p P(p; a, b), \quad (20)$$

where $P(a, b)$ is the probability that the links a and b are connected by a percolation hull. Every hull p that joins

these links contributes to the last expression above with its own probability $P(p; a, b)$.

We see that the geometric objects that satisfy restriction with respect to absorbing boundaries for the network model of the SQH transition are percolation hulls. At the critical point and in the continuum limit these hulls become SLE_6 lines (this is rigorously known for the site percolation on triangular lattice and its variants,^{64–66} and is believed to be true for other percolation models). In relation to conformal restriction, SLE_6 was studied in Ref. [43] where it was shown that chordal SLE_6 conditioned not to intersect the real line satisfies two-sided conformal restriction with exponent $h_A = 1$. It then follows that the right boundary of such conditioned SLE_6 is $\text{SLE}(8/3, \rho_A)$ with $\rho_A = 2/3$. Furthermore, SLE_6 conditioned not to intersect the positive half-line satisfies the one-sided conformal restriction with the exponent $h = 1/3$. Its right boundary is $\text{SLE}(8/3, -2/3)$.

As we will see below in Section VI, the microscopic picture of the SQH critical point as the critical bond percolation on a square lattice allows us to identify the boundary operators in the corresponding CFT, and, consequently, obtain exact results for various correlation functions of these operators. Physically, these correlation functions are average PCCs in the presence of various complicated boundary conditions. In the case of SQH transition they are known explicitly, including values of conformal dimensions. In particular, we will show that $h_{LA} = 1/3$ and $h_{RA} = h_A = 1$ ($\rho_{LA} = -2/3$, $\rho_{RA} = \rho_A = 2/3$). However, the main point of this paper is that in all systems that we consider the same exact results (including non-trivial spatial dependence of PCCs) are valid, except that the values of the two exponents h_{RA} and h_{LA} are not always known exactly.

C. Classical limit of the Chalker-Coddington model

The two network models considered so far both had critical points separating insulating states. In this section we consider a classical variant of the CC network model which leads to a critical (metallic) behavior for any values of parameters. This classical model describes diffusive transport of electrons in high magnetic fields. The model has been studied in detail in Ref. [56]. Here we briefly summarize results of this analysis.

The classical limit of the (isotropic) CC model is obtained if we neglect quantum interference effects. Thus, we consider a classical particle performing a random walk along the links of the network. Every time the walker approaches a node, it turns right with probability $R = t_A^2$ or left with probability $T = 1 - t_A^2$. Notice that in this limit the model becomes non-random since the (random) phases that were present on the links in the quantum CC model are not considered any more. Observables in this model are not random quantities then, and we do not need to average them. For example, the PCC $g(a, b)$ in this model is simply given by the probability that a ran-

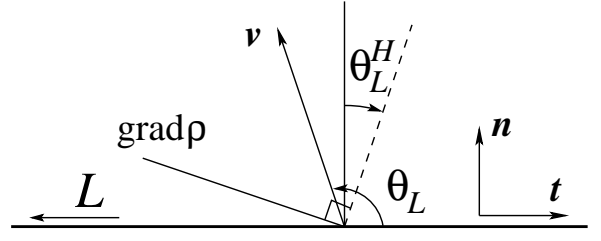


FIG. 8: Various directions and angles at a “left” boundary. Here $T = 1/4$, $R = 3/4$, so that $\gamma = \tan \theta_L^H = -1/3$.

dom walker starting at the link a reaches link b for the first time without returning to a . Thus

$$g(a, b) = \sum_p P(p; a, b), \quad (21)$$

where $P(p; a, b)$ is the probability for a random walker to follow a particular path p between the links a and b . These probabilities are easily seen to satisfy restriction with respect to absorbing boundaries.

In an infinite system the time evolution of probability $\rho(\mathbf{r}, t)$ for a random walker to be at point \mathbf{r} at time t can be obtained by the Fourier transform. In the long-wave limit this results in a diffusive spectrum

$$-i\omega_k = Dk^2, \quad D = \frac{a^2}{4\tau} \frac{RT}{R^2 + T^2}, \quad (22)$$

where a is the lattice spacing, and τ is the time step. This leads to the diffusive behavior of the coarse-grained probability density $\bar{\rho}$ in the continuum limit:

$$\partial_t \rho(\mathbf{r}, t) = D \nabla^2 \rho(\mathbf{r}, t). \quad (23)$$

The corresponding probability current (which can be microscopically defined in various equivalent ways) is

$$\begin{aligned} j_x(\mathbf{r}, t) &= -(\sigma_{xx}^0 \partial_x + \sigma_{xy}^0 \partial_y) \rho(\mathbf{r}, t), \\ j_y(\mathbf{r}, t) &= -(\sigma_{yx}^0 \partial_x + \sigma_{yy}^0 \partial_y) \rho(\mathbf{r}, t), \end{aligned} \quad (24)$$

where

$$\sigma_{xx}^0 = \sigma_{yy}^0 = \frac{RT}{R^2 + T^2}, \quad \sigma_{xy}^0 = -\sigma_{yx}^0 = -\frac{T^2}{R^2 + T^2} \quad (25)$$

are the components of the classical conductivity tensor.

As in the cases considered above, in a system with boundaries one can distinguish three types of boundary conditions: absorbing (corresponding to ideal leads) and “left” and “right” reflecting (hard wall), see their definitions at the end of Sec. II. At absorbing boundaries the probability density vanishes:

$$\rho = 0 \quad \text{on absorbing boundaries}, \quad (26)$$

since a random walker that hits an absorbing boundary

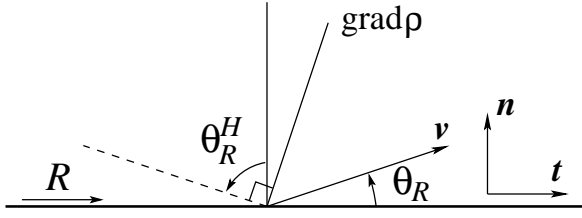


FIG. 9: Various directions and angles at a “right” boundary. Here $T = 1/4$, $R = 3/4$, so that $\tan \theta_R^H = -1/\gamma = 3$.

escapes to the lead and never returns to the system.

On the other hand, at a reflecting boundary one must impose the vanishing of the normal component j_n of the current. In Ref. [56] the authors only considered what we call a “left” reflecting boundary. In this case the requirement that $j_n = 0$ leads in the continuum to the following boundary condition:

$$(\partial_n - \gamma \partial_t) \rho = 0, \quad (27)$$

where we have introduced the notation

$$\gamma = \frac{\sigma_{xy}^0}{\sigma_{xx}^0} = -\frac{T}{R}. \quad (28)$$

It is useful to depict this boundary condition by drawing a straight line orthogonal to the direction of the gradient of the density at the boundary. In Fig. 8 we show a classical Hall system occupying the upper half plane. We assume that the boundary of the system (along the x axis) is a “left” boundary. Then, as follows from Eq. (27), the line orthogonal to $\nabla \rho$ is in the direction (T, R) . This direction of the vanishing component of the density gradient is shown as a dashed line, together with the direction of the gradient itself (without an arrow). We define the Hall angle θ_L^H as the angle that the dashed line makes with the inward normal. Then we have

$$\tan \theta_L^H = \gamma. \quad (29)$$

Notice that the sign in this expression is consistent, since the angle θ_H is negative in the usual sense (we go clockwise from the normal to the dashed line).

Similarly, at a “right” reflecting boundary the vanishing of j_n leads to

$$\left(\partial_n + \frac{1}{\gamma} \partial_t\right) \rho = 0. \quad (30)$$

We now depict this boundary condition at a “right” boundary of the Hall system occupying the upper half plane, see Fig. 9. The gradient of the density in this case is parallel to the vector (T, R) . The direction in which the component of $\nabla \rho$ vanishes is then $(-R, T)$. We show both these directions in Fig. 9. The Hall angle defined

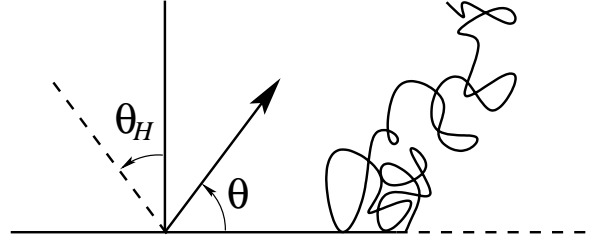


FIG. 10: The Hall angle θ_H is measured from the normal to the reflecting boundary (solid line). The angle θ measured from the positive direction on the real axis is the angle of reflection of a Brownian motion started at the boundary point where the reflecting boundary switches to the absorbing boundary (dashed line).

as before, now satisfies

$$\tan \theta_R^H = \frac{R}{T} = -\frac{1}{\gamma} = -\cot \theta_L^H. \quad (31)$$

It follows then that the Hall angles at the two types of reflecting boundaries are related by

$$\theta_R^H - \theta_L^H = \frac{\pi}{2}. \quad (32)$$

A steady (time independent) density and current profiles in the continuum limit are governed by the Laplace equation $\nabla^2 \rho = 0$ supplemented by the boundary conditions (26, 27, 30) on the two types of boundaries. There is an alternative (and equivalent) description of the continuum limit in terms of reflected Brownian excursions. These excursions are continuum limits of random walks on the network, which are conditioned not to hit absorbing boundaries, and which are reflected in the direction $e^{i\theta}$ upon hitting a reflecting boundary placed along the horizontal (real) axis, see Fig. 10.

This direction can be obtained microscopically from the classical CC model as follows. Imagine a random walker at the “left” boundary shown in the bottom on Fig. 5. Then in one time step the walker moves by the distance $2a$ (a is the lattice spacing between the middle points of the links of the network) along the boundary to the left with probability T (left turn) or normal to the boundary with probability R (right turn). Then the expected displacement of the walker after one time step from the boundary is $2a(R\mathbf{n} - T\mathbf{t})$. This displacement is in the direction of the vector $\mathbf{v} \propto (-T, R)$ shown in Fig. 8. We denote by θ_L the angle the vector \mathbf{v} makes with the tangent vector \mathbf{t} . Notice that the direction of \mathbf{v} is related to the direction of the dashed line by a reflection across the normal to the boundary, so we have

$$\theta_L = \frac{\pi}{2} - \theta_L^H. \quad (33)$$

Similarly, on a “right” reflecting boundary we have the

relation

$$\theta_R = \frac{\pi}{2} - \theta_R^H, \quad (34)$$

see Fig. 9. This is, actually, a general relation between the direction of reflection and the direction along which the gradient vanishes, as can be inferred, for example, from Dubedat's paper on reflected BMs.⁶⁷ As a consequence of Eq. (32) the angles of reflection at the two types of reflecting boundaries are related by

$$\theta_L - \theta_R = \frac{\pi}{2}. \quad (35)$$

Brownian excursions reflected at an angle θ from a part of the boundary are known to satisfy conformal restriction property with the (one-sided) restriction exponent^{43,44}

$$h = 1 - \frac{\theta}{\pi}. \quad (36)$$

In terms of the Hall angles $\theta_{R,L}^H$ we get the two possible values of the one-sided restriction exponents corresponding to the two possible types of reflecting boundaries:

$$h_{RA} = \frac{1}{2} + \frac{\theta_R^H}{\pi}, \quad h_{LA} = \frac{1}{2} + \frac{\theta_L^H}{\pi}. \quad (37)$$

Then the relation (32) implies the following:

$$h_{RA} - h_{LA} = \frac{1}{2}. \quad (38)$$

While this relation holds for the classical limit of the CC model (classical diffusion in magnetic field), a priori it is not valid for other systems we consider in this paper. For example, for the SQH transition the exponents are known exactly to be $h_{RA} = 1$, $h_{LA} = 1/3$, see Section VI, so they do not satisfy the relation (38).

The diffusive behavior in a magnetic field can also be described by a simple Gaussian theory⁵⁶ of a complex scalar field $z(\mathbf{r})$ with the action

$$S_0 = \frac{\sigma_{xx}^0}{4} \int d^2r \partial_\mu z \partial_\mu \bar{z} + \frac{\sigma_{xy}^0}{4} \int d^2r \epsilon_{\mu\nu} \partial_\mu z \partial_\nu \bar{z}, \quad (39)$$

where \bar{z} is the complex conjugate of z (and we use the convention where field configurations are weighted by e^{-S_0} in the functional integral). The propagator of the field z

$$d(\mathbf{r}, \mathbf{r}') = \frac{\sigma_{xx}^0}{4} \langle \bar{z}(\mathbf{r}) z(\mathbf{r}') \rangle_0, \quad (40)$$

where $\langle \dots \rangle_0$ stands for the average in the field theory with the action S_0 , satisfies (with respect to the coordinate \mathbf{r}) the same equations and boundary conditions as the density ρ above. Its relation to transport properties of the diffusive system are described in detail in Ref. [56].

D. Metal in class D

At the mean field level, the problem of thermal transport of a disordered superconductor with broken spin rotation and time reversal symmetries belongs to class D in the AZ classification.⁵⁷⁻⁵⁹ A generic model in this class can have an insulating, a thermal quantum Hall, and a metallic state. The metallic state at weak disorder can be described by a nonlinear sigma model. We will be interested in the weak coupling regime of this model where perturbation theory is justified, and the replica and the supersymmetry formulations give the same results. In the compact replica formulation the sigma model action is⁵⁷

$$S = \frac{\sigma_{xx}^0}{8} \int d^2r \text{tr} \partial_\mu Q \partial_\mu Q + \frac{\sigma_{xy}^0}{8} \int d^2r \text{tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q, \quad (41)$$

where Q is a $2n \times 2n$ matrix from the coset $O(2n)/U(n)$, and σ_{xx}^0 and σ_{xy}^0 are bare longitudinal and Hall thermal conductivities in a certain normalization. A possible parametrization for the sigma model field Q is

$$Q = \begin{pmatrix} \sqrt{1 - ZZ^\dagger} & Z \\ Z^\dagger & -\sqrt{1 - Z^\dagger Z} \end{pmatrix}, \quad (42)$$

where Z is a complex antisymmetric matrix.

When the bare σ_{xx}^0 is large, we can treat the sigma model perturbatively. The Hall conductivity is not renormalized perturbatively. At the same time, at one loop the diagonal conductivity is renormalized (with increasing system size L) as

$$\frac{d\sigma_{xx}}{d \ln L} = \frac{1}{2\pi}, \quad (43)$$

so at sufficiently large scale L the conductivity is logarithmically large $\sigma_{xx}(L) \sim \log L$. If we consider such a large metallic system in class D, then the leading (in inverse powers of $\sigma_{xx}(L)$) behavior of correlation functions (including transport properties) of the system will be described by the first nontrivial (quadratic) term of the expansion of the action (41) in powers of the matrix Z that appears in Eq. (42). In terms of the matrix elements z_{ij} of this matrix this quadratic term is simply

$$S_0[Z] = \frac{\sigma_{xx}(L)}{4} \int d^2r \partial_\mu z_{ij} \partial_\mu \bar{z}_{ij} + \frac{\sigma_{xy}^0}{4} \int d^2r \epsilon_{\mu\nu} \partial_\mu z_{ij} \partial_\nu \bar{z}_{ij}. \quad (44)$$

This is, essentially, the same action as S_0 (39) except that there are $n(n-1)$ copies of the complex field z .

We conclude that the leading transport behavior of metal in class D is the same as diffusion in magnetic field described in the previous section, and can be alternatively described by conformal restriction theory. The values of the Hall conductivity σ_{xy}^0 is more or less ar-

bitrary, and therefore, in this case we again deal with arbitrary values of the Hall angles $\theta_{L,R} \in [-\pi/2, \pi/2]$, and the one-sided restriction exponents $h_{L,R} \in [0, 1]$.

We comment here that there are different network models in class D, that have been studied numerically.^{23,26} In one of these models, the so called O(1) model, the random phases on the links can only take values ± 1 independently. The model appears to have only the metallic phase. It is natural to ask whether one can identify proper geometric objects and establish the restriction property directly at the level of the O(1) network model, similar to how it has been done for the CC model above. The same question exists for other network models in class D, including the Cho-Fisher model⁶⁸ and the network model equivalent to the Ising spin glass.⁶² At present this remains an interesting open problem. We note, however, that a straightforward generalization of the averaging over the phases in the CC model (see Eq. (15)) to the O(1) model leads to objects (analogues of the pictures in the CC model) where the numbers of the advanced and retarded paths on a given link have the same parity. It is clear then that if we truncate this description by retaining only the objects which include links visited by either retarded or advanced paths at most once, we obtain exactly the truncated model of Ikhlef et al.⁶³ This illustrates the drastic nature of the truncation procedure which leads to the same model starting from networks in different symmetry classes.

IV. THE THEORY OF CONFORMAL RESTRICTION

In this section we review basic properties of conformal restriction measures and how they are related to multiple SLE(8/3, ρ).

A. The basic theorem of conformal restriction

The basic theorem of conformal restriction states that the statistics of a restriction measure is determined by a single real parameter h called the restriction exponent.⁴³ The proof is based on the idea that the statistics of a restriction measure in a domain D is fully determined by the collection of probabilities $P_A \equiv P[F \in D \setminus A]$ that a sample F from this measure avoids an arbitrary subset A attached to the boundary of D , as described in Section II A. One then proceeds to show that P_A is uniquely determined by a single parameter h that shall be called the restriction exponent. In fact, for the case where the restriction measure lies in the upper half plane \mathbb{H} , is anchored at the origin and aims at infinity, it was proved that P_A is given by the following expression:

$$P_A = |\Phi'_A(0)|^h, \quad (45)$$

where Φ_A is the conformal map from $\mathbb{H} \setminus A$ to \mathbb{H} , which fixes the origin and infinity $\Phi_A(0) = 0$, $\Phi_A(\infty) = \infty$ and has unit derivative at infinity $\Phi'_A(\infty) = 1$.

To prove this assertion, one proceeds as follows. Take two sets A and B and consider their union $A \cup B$. The probability $P_{A \cup B}$ that a restriction measure avoids $A \cup B$ can be written as $P_{A \cup B} = P_A P_{B|A}$, where the second factor on the RHS is the conditional probability to avoid B given that A is avoided. Now, the conformal restriction property, assumed to hold, means that this second factor can be written as $P_{B|A} = P_{\Phi_A(B)}$, where we have employed the map Φ_A to “remove” the avoided set A . Thus, we get the functional equation

$$P_{A \cup B} = P_A P_{\Phi_A(B)} \quad (46)$$

The rest of the proof consists of solving this equation. In order to appreciate the equation’s structure we switch notations and instead of labeling the probability P_A by the set A we label it by the map Φ_A and write $P(\Phi_A)$. Now $A \cup B$ is associated with the map $\Phi_{A \cup B} = \Phi_{\Phi_A(B)} \circ \Phi_A$. The functional equation (46) takes the form

$$P(\Phi_{\Phi_A(B)} \circ \Phi_A) = P(\Phi_A) P(\Phi_A(B)). \quad (47)$$

Thus, P maps the composition operation in the space of conformal maps into simple multiplication. It is clear that (45) obeys (47) as the factor $|\Phi'_A(0)|$ is the Jacobian of the transformation Φ_A at the origin, and as such get multiplied as successive maps are composed. One can show that, in fact, $|\Phi'_A(0)|$ is the only solution up to the arbitrary parameter h .⁴³

From the above argument it is easy to deduce that the total weights (partition functions) $Z_{D_1}(a, b)$ and $Z_{D_2}(c, d)$ of restriction measures with the same restriction exponent h defined in two different domains D_1 and D_2 satisfy the following transformation property:

$$Z_{D_1}(a, b) = |f'(a)|^h |f'(b)|^h Z_{D_2}(f(a), f(b)), \quad (48)$$

where $f(z)$ is a conformal map from D_1 to D_2 that maps the marked points a and b to $f(a) = c$ and $f(b) = d$. This transformation law is that of a two-point correlation function of *primary* operators, in terminology of CFT.

The above description of a restriction measure by the probabilities of avoidance of certain sets given by (45) is somewhat implicit. To obtain a more explicit description of restriction measures which will be also useful for computations we can concentrate on the boundaries of a restriction sample (a cluster). These cluster boundaries are known to be variants of SLE_{8/3} distorted in a particular way, and are described by a variant of the Schramm-Loewner equation. In particular, the ensemble of the SLE_{8/3} curves satisfies the two-sided conformal restriction with the restriction exponent $h = 5/8$.⁴³

In the next subsection we describe the SLE method for critical curves.

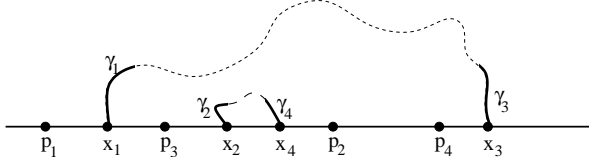


FIG. 11: Two SLE's interacting with four points p_i in the upper half plane $D = \mathbb{H}$.

B. Description of critical curves through SLE

Suppose we have some statistical system defined in a domain D in which $2n + m$, x_i , $i = 1, \dots, 2n$ and p_j , $j = 1, \dots, m$ special points are marked on the boundary. Figure 11 illustrates this situation for a system in the upper half plane, so that the marked points lie on the real axis. The partition function of such a system is denoted by $Z_D(x_1, \dots, x_{2n}; p_1, \dots, p_m)$. The points x_i are beginnings and ends of curves that are created in the statistical system. Namely, we assume that there exist n curves in the upper half plane, the i 'th curve connecting x_i with x_{n+i} on the real axis. p_j denote points where boundary conditions are changed or operators are inserted, etc. The shape of the curves depends on the positions of the m marked points p_j in some way. We shall consider only the case where the operators inserted at p_j are primary fields of weights h_j . It turns out that the points x_i also correspond to insertions of primary fields, and that all these have the same weight $h_\kappa = (6 - \kappa)/(2\kappa)$. Then the partition function Z_D transforms under conformal maps of the domain D and the marked points on its boundary as a correlation function of primary operators in CFT:

$$Z_D(x_1, \dots, x_{2n}; p_1, \dots, p_m) = \prod_{i=1}^{2n} |g'(x_i)|^{h_\kappa} \prod_{j=1}^m |g'(p_j)|^{h_j} \times Z_{\tilde{D}}[g(x_1), \dots, g(x_{2n}); g(p_1), \dots, g(p_m)], \quad (49)$$

where g is a conformal map from D to \tilde{D} .

Furthermore, we require the partition function $Z_{\mathbb{H}}$ in the upper half plane \mathbb{H} to be invariant with respect to global conformal transformations that preserve \mathbb{H} : translations, scaling, and special conformal transformations. These transformations form the $SL(2, \mathbb{R})$ group and are generated by

$$\begin{aligned} L_{-1} &= \sum_{i=1}^{2n} \partial_{x_i} + \sum_{j=1}^m \partial_{p_j}, \\ L_0 &= \sum_{i=1}^{2n} (x_i \partial_{x_i} + h_i) + \sum_{j=1}^m (p_j \partial_{p_j} + h_\kappa), \\ L_{+1} &= \sum_{i=1}^{2n} (x_i^2 \partial_{x_i} + 2h_i x_i) + \sum_{j=1}^m (p_j^2 \partial_{p_j} + 2h_\kappa p_j). \end{aligned} \quad (50)$$

Thus we should have

$$L_{\pm 1,0} Z_{\mathbb{H}} = 0. \quad (51)$$

Let us now mark $2n$ disjoint segments on the curves, γ_i , $i = 1, \dots, 2n$, so that the i 'th segment connects the point x_i to a point z_i on the i -th curve for $i = 1, \dots, n$ or the $i - n$ -th curve for $i = n + 1, \dots, 2n$ (see figure (11)). Let us also denote the contribution to the partition function from configurations of the statistical system in which the segments take a particular shape γ_i as $Z(\gamma_1, \dots, \gamma_{2n}; p_1, \dots, p_m)$. With the shape of the curves γ_i fixed, the quantity $Z(\gamma_1, \dots, \gamma_{2n}; p_1, \dots, p_m)$ is the partition function in the *slit domain* $\mathbb{H} \setminus \bigcup_{i=1}^{2n} \gamma_i$, that is, the upper half plane with the segments γ_i removed. The full partition function is obtained by summing these partial contributions over the shapes of all segments γ_i . Somewhat abusing notation for the sum over all possible shapes of curves γ_i we write

$$\begin{aligned} Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) \\ = \sum_{\{\gamma_i\}} Z(\gamma_1, \dots, \gamma_{2n}; p_1, \dots, p_m). \end{aligned} \quad (52)$$

The points z_i sit at the tips of the segments. We can enlarge or reduce the size of the segments by moving the tips z_i along the curves. We shall parameterize this motion along the i -th curve by a "time" parameter t_i . To describe the shape of the segments at arbitrary times t_i SLE makes use of a conformal map from the slit domain $\mathbb{H} \setminus \bigcup_{i=1}^{2n} \gamma_i$ back to the upper half plane. This conformal map $g_{\mathbf{t}}(z)$ depends on all the times t_i which together form the vector \mathbf{t} . This time dependence is given by a stochastic differential equation which determines $g_{\mathbf{t}}(z)$ as a function of t_i :

$$d_{t_i} g_{\mathbf{t}}(z) = \frac{2dt_i}{g_{\mathbf{t}}(z) - \xi_i(\mathbf{t})}. \quad (53)$$

Here $\xi_i(\mathbf{t}) = g_{\mathbf{t}}(z_i)$ are the images of the tips of the segments γ_i under the map $g_{\mathbf{t}}(z)$. Each ξ_i is located on the real axis and obeys the same equation (53) with respect to t_j for $j \neq i$. However for $z = z_i$ Eq. (53) is ill-defined and must be replaced by:

$$\begin{aligned} d_{t_i} \xi_i &= \kappa \partial_{\xi_i} \log Z_{\mathbb{H}}(\xi_1, \dots, \xi_{2n}; g_{\mathbf{t}}(p_1), \dots, g_{\mathbf{t}}(p_m)) dt_i \\ &+ \sqrt{\kappa} dB_i, \end{aligned} \quad (54)$$

where $B_i(t)$ are independent standard Brownian motions. When all the times are increased by small increments dt_i , the evolution of the conformal map $g_{\mathbf{t}}(z)$ can be described

by the following system of equations:

$$dg_t(z) = \sum_{i=1}^{2n} \frac{2dt_i}{g_t(z) - \xi_i(t)}, \quad (55)$$

$$d\xi_i = \kappa \partial_{\xi_i} \log Z_{\mathbb{H}}(\xi_1, \dots, \xi_{2n}; g_t(p_1), \dots, g_t(p_m)) dt_i + \sqrt{\kappa} dB_i + \sum_{j \neq i} \frac{2dt_j}{\xi_i - \xi_j}. \quad (56)$$

These equations are essentially the same as Eqs. (2, 3) in Ref. [45]. Note that Eq. (55) fixes the asymptotic behavior of the map $g_t(z) \sim z + (\sum_i t_i)/z$ as $z \rightarrow \infty$. The $2n$ curves whose evolution is described by Eqs. (55, 56) are often called SLE traces or simply traces. Since the equations are stochastic differential equations they, in fact, define a probability measure on the traces, or an ensemble.

So far we did not specify the partition function $Z_{\mathbb{H}}$ that appears in the general SLE equation (56). In the next subsection we consider special cases where $Z_{\mathbb{H}}$ will be uniquely determined by the global conformal invariance (51). For the special value $\kappa = 8/3$ this will give a description of general one-sided restriction measures.

C. Description of restriction measures using SLE(8/3, ρ)

One can obtain a restriction measure by taking $m = 0$, $n = 1$ and $x_2 \rightarrow \infty$. By translational invariance ($L_{-1}Z_{\mathbb{H}} = 0$) we immediately obtain that $Z_{\mathbb{H}}(x_1, \infty)$ is a constant and thus the forcing term in (54) vanishes. We shall also drop t_2 , never considering the evolution with respect to this time. The equations (53) and (54) reduce to:

$$dg_t(z) = \frac{2dt}{g_t(z) - \xi(t)}, \quad d\xi = \sqrt{\kappa} dB_t. \quad (57)$$

Different values of κ lead to different ensembles of curves. The only value of κ for which the restriction property holds is $\kappa = 8/3$.⁴³ The restriction exponent in this case turns out to be $h = 5/8$. This corresponds to the insertion at x_1 of a primary operator of weight $h = 5/8$.

Thus, the SLE process without forcing thus can only produce a restriction measure of a single exponent $5/8$. It is possible to modify the SLE_{8/3} process somewhat and obtain more general one-sided restriction measures with other restriction exponents. We shall still be interested in the case where a single curve emanates from x_1 and ends at infinity. In order to be able to generalize the usual SLE procedure which gives rise to the exponent $5/8$ one has to employ a point splitting procedure. Namely, we shall take the point x_1 and replace it by two points x_1 and $X_1 < x_1$ (this choice will lead to a restriction measure in the upper half plane whose right boundary will be the SLE(8/3, ρ) curve). The SLE trace shall emanate from x_1 and X_1 will be a marked point (previously this was

denoted by p_1). At the end of the procedure the two points are re-fused, and we have a trace emanating from a single point x_1 which is also a marked point. Note however, that independently of whether the points x_1 and X_1 are fused or not, for any non-zero time $t = t_1$ the partition function appearing in (54) will depend on three distinct points $p_1 = g_t(X_1)$, $\xi = g_t(z_1)$ and ∞ . X_1 will correspond as usual to an insertion of a primary operator. The global conformal invariance determines in this case partition function to have a simple power law dependence whose exponent is denoted by $3\rho/8$:

$$Z_{\mathbb{H}}(\xi, \infty, p_1) = \frac{1}{(\xi - p_1)^{3\rho/8}}. \quad (58)$$

With this form of the partition function the general equations (55, 56) reduce to

$$dg_t(z) = \frac{2dt}{g_t(z) - \xi(t)}, \quad d\xi = \frac{\rho dt}{\xi - g_t(X_1)} + \sqrt{\kappa} dB_t. \quad (59)$$

The value of ρ determines the restriction exponent:

$$h = \frac{(3\rho + 10)(2 + \rho)}{32}. \quad (60)$$

Moreover it determines the conformal weight of the operator at X_1 , which is given by

$$h' = \frac{\rho(4 + 3\rho)}{32}. \quad (61)$$

The process one obtains in this way is termed SLE(8/3, ρ).

The described procedure shows that boundaries of one sided restriction measures can be created by fusing the operator creating an SLE_{8/3} curve with a primary operator. Since a restriction measure is fully specified by its exponent, by appropriately choosing the parameter $\rho > -2$ in the above procedure, we can obtain restriction measures with any given exponent $h > 0$. We shall assume in the following that this also holds locally, independently of other curves or marked points. This assumption does not have a rigorous mathematical proof. It rests on the physical assumption that any additional operators or curves can only change the large scale properties of the given curve rather than its local structure.

D. Martingale conditions on partition functions

When considering more general situations where restriction holds, we shall have to consider more complicated partition functions than those appearing in the previous section IV C. To compute those partition functions it is not sufficient to use the conformal covariance (49) and the global SL(2, \mathbb{R}) invariance (51). In addition to these conditions we must make use of the idea of the par-

tial summation over segments of curves expressed by Eq. (52), to obtain further conditions on $Z_{\mathbb{H}}$.

We will derive these well known conditions in this section. In order to set notation, we first consider the situation where $m = 0$ and the forcing term in (54) vanishes, namely

$$d_{t_i}\xi_i = \sqrt{\kappa}dB_i. \quad (62)$$

In this situation $2n$ independent curves start at the points x_i and go to infinity. Due to independence of the Brownian motions in (62), the statistical weight of the SLE traces in this case is given by

$$\prod_{i=1}^{2n} Z(\gamma_i, \infty). \quad (63)$$

This product of independent SLE measures can be used to define expectation values by

$$\mathbb{E}[f(\gamma_1, \dots, \gamma_{2n})] \equiv \sum_{\{\gamma_i\}} \left[f(\gamma_1, \dots, \gamma_{2n}) \prod_{i=1}^{2n} Z(\gamma_i, \infty) \right], \quad (64)$$

where the sum over γ_i is used in the same sense as in Eq.

(52) and denotes summing over all possible shapes of the curves.

We now consider the general case with a non-trivial forcing term in Eq. (54). The SLE measure in this case is different from the product (63). However, it is possible to describe a general ensemble of SLE curves using the product measure (63) produced by independent Brownian motions (62). Heuristically we can think of the measure (63) produced by (62) serving to scan a large class of curves. Very loosely speaking, we can consider curves sampled from the non-interacting measure (63) and *the same* curves from the general SLE ensemble, and compare their weights. Alternatively, we can think of creating the probability measure in (54) by first producing the curves using (62) and then re-weighting them. A priori it is not at all clear that this should be possible, since the weights of the curves sampled from the two measures can be incomparable (for example, one weight can be zero, while the other non-zero). However, it happens to be possible in the case of the two SLE measures with the same value of κ : one with the independent forcing (62) and the other with a non-trivial forcing (54).⁶⁹ More formally, for an expectation value in a general SLE ensemble we have:

$$\begin{aligned} \sum_{\{\gamma_i\}} Z(\gamma_1, \dots, \gamma_{2n}; p_1, \dots, p_m) f(\gamma_1, \dots, \gamma_{2n}) &= \sum_{\{\gamma_i\}} \frac{Z(\gamma_1, \dots, \gamma_{2n}; p_1, \dots, p_m) f(\gamma_1, \dots, \gamma_{2n})}{\prod_{i=1}^{2n} Z(\gamma_i, \infty)} \prod_{i=1}^{2n} Z(\gamma_i, \infty) \\ &= \mathbb{E} \left[\frac{Z(\gamma_1, \dots, \gamma_{2n}; p_1, \dots, p_m) f(\gamma_1, \dots, \gamma_{2n})}{\prod_{i=1}^{2n} Z(\gamma_i, \infty)} \right], \end{aligned} \quad (65)$$

where \mathbb{E} is the one defined in (64), that is the expectation value with respect to the measure of the independent SLE's. This method of re-weighting the measure, when done with appropriate mathematical rigor is known under the name of Girsanov's transformation. The condition that the weights of the curves sampled from two measures are comparable, translates in more rigorous terms to the condition of absolute continuity of one measure with respect to the other. (For Girsanov's theorem and other information on stochastic analysis see, for example, Refs. [70, 71].)

We now make use of the re-weighting procedure to obtain further conditions on $Z_{\mathbb{H}}$. We freeze all the times t_i except t_1 and rename it $t_1 = t$. Consider the partition function $Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m)$. By definition it is independent of t :

$$\partial_t Z_{\mathbb{H}}(p_1, \dots, p_m; x_1, \dots, x_{2n}) = 0. \quad (66)$$

When t is varied, only the segment γ_1 is produced, and the partial summation (52) can be used in the form

$$Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) = \sum_{\gamma_1} Z(\gamma_1, x_2, \dots, x_{2n}; p_1, \dots, p_m). \quad (67)$$

Now we can use the re-weighting procedure (65) with $f = 1$:

$$Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) = \mathbb{E} \left[\frac{Z(\gamma_1, x_2, \dots, x_{2n}; p_1, \dots, p_m)}{Z(\gamma_1, \infty)} \right]. \quad (68)$$

Next we transform both the numerator and the denominator in the last expression using the SLE map from $\mathbb{H} \setminus \gamma_1$ to

\mathbb{H} and recalling the covariance property (49):

$$Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) = \mathbb{E} \left[\prod_{i=2}^{2n} |g'_t(x_i)|^{h_\kappa} \prod_{j=1}^m |g'_t(p_j)|^{h_j} Z_{\mathbb{H}}[\xi_1(t), g_t(x_2), \dots, g_t(x_{2n}); g_t(p_1), \dots, g_t(p_m)] \right]. \quad (69)$$

Notice that the singular derivative $g'_t(x_1)$ has canceled between the numerator and the denominator. The last equation is valid at any time t . We now substitute this into Eq. (66):

$$\partial_t \mathbb{E} \left[\prod_{i=2}^{2n} |g'_t(x_i)|^{h_\kappa} \prod_{j=1}^m |g'_t(p_j)|^{h_j} Z_{\mathbb{H}}[\xi_1(t), g_t(x_2), \dots, g_t(x_{2n}); g_t(p_1), \dots, g_t(p_m)] \right] = 0. \quad (70)$$

Conditions of this type often appear in stochastic analysis when one studies special types of stochastic processes called martingales.^{70,71} Roughly speaking, a martingale is a stochastic process $M(t)$ whose expectation value is constant in time: $\partial_t \mathbb{E}[M(t)] = 0$. For this reason we call consequences of Eq. (70) derived below martingale conditions on partition functions.

Now we set $t = 0$ in (70). It is then straightforward to use the stochastic equations (53) and (62) and Ito's formula^{70,71} to transform this equation into a Fokker-Planck equation:

$$\left[\frac{\kappa}{2} \partial_{x_1}^2 - 2 \sum_{i=2}^{2n} \left(\frac{h_\kappa}{(x_i - x_1)^2} - \frac{1}{x_i - x_1} \partial_{x_i} \right) - 2 \sum_{j=1}^m \left(\frac{h_j}{(p_j - x_1)^2} - \frac{1}{p_j - x_1} \partial_{p_j} \right) \right] Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) = 0. \quad (71)$$

This equation was derived using evolution with respect to t_1 only. Similar equations result when we use other times t_i , so in the end we get $2n$ martingale conditions

$$\left[\frac{\kappa}{2} \partial_{x_i}^2 - 2 \sum_{j \neq i}^{2n} \left(\frac{h_\kappa}{(x_j - x_i)^2} - \frac{1}{x_j - x_i} \partial_{x_j} \right) - 2 \sum_{k=1}^m \left(\frac{h_k}{(p_k - x_i)^2} - \frac{1}{p_k - x_i} \partial_{p_k} \right) \right] Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) = 0, \quad (72)$$

which the partition function $Z_{\mathbb{H}}$ must satisfy in addition to the conditions of conformal invariance (51).

The Fokker-Planck equations can be written in a compact way using the following differential operators:

$$\mathcal{L}_{-m}(x_i) = \sum_{j \neq i}^{2n} \left(\frac{(m-1)h_\kappa}{(x_j - x_i)^m} - \frac{1}{(x_j - x_i)^{m-1}} \partial_{x_j} \right) + \sum_{k=1}^m \left(\frac{(m-1)h_k}{(p_k - x_i)^m} - \frac{1}{(p_k - x_i)^{m-1}} \partial_{p_k} \right). \quad (73)$$

Notice that each operator insertion except x_i contributes a term to $\mathcal{L}_{-m}(x_i)$. The translational invariance condition $L_{-1}Z_{\mathbb{H}} = 0$ can be written as

$$\partial_{x_i} Z_{\mathbb{H}} = \mathcal{L}_{-1}(x_i) Z_{\mathbb{H}}. \quad (74)$$

Thus, trading the derivative ∂_{x_i} for $\mathcal{L}_{-1}(x_i)$, the Fokker-Planck equations (72) can be written as

$$\left(\frac{\kappa}{2} \mathcal{L}_{-1}^2(x_i) - 2 \mathcal{L}_{-2}(x_i) \right) Z_{\mathbb{H}}(x_1, \dots, x_{2n}; p_1, \dots, p_m) = 0. \quad (75)$$

V. CONFORMAL RESTRICTION AND CFT IN THE COULOMB GAS FORMALISM

In the following Section VI we will make use of conformal restriction to obtain certain information on the transport behavior of the system. The development can

be cast solely in the language of conformal restriction, however, we shall make use of ideas which are already well developed in the language of CFT, so we prefer to mix the two approaches in the presentation. In this section we set up notations related to CFT in the so called the Coulomb gas formalism.⁷² To find more details the reader may consult the references [29,40–42].

As we have already mentioned, from a field theory perspective restriction models are described by conformal field theory (CFT) with vanishing central charge, $c = 0$. In the Coulomb gas formalism, correlations function of CFT are computed by a certain ansatz. The CFT correlation function is replaced by a correlation function for the Gaussian free field, with a possible introduction of certain additional non-local operators, called screening charges. We shall describe the procedure briefly below.

Consider a Gaussian free field, namely a fluctuating

bosonic field with the action:

$$S = \frac{1}{8\pi} \int d^2r (\nabla\varphi)^2. \quad (76)$$

This action describes a CFT with $c = 1$. We can modify the central charge to $c < 1$ by introducing the so called background charge $-2\alpha_0$, where α_0 is related to the central charge as follows:

$$c = 1 - 24\alpha_0^2. \quad (77)$$

Next, consider vertex operators, namely, the exponentials of the free field

$$V_\alpha(z, \bar{z}) = e^{i\sqrt{2}\alpha\varphi(z, \bar{z})}, \quad (78)$$

where the parameter α is called the Coulomb charge of the operator. In the CFT with the background charge the vertex operator V_α has the conformal weight

$$h(\alpha) = \alpha(\alpha - 2\alpha_0). \quad (79)$$

The last ingredient we need are the screening operators defined as:

$$Q_\pm = \int dz d\bar{z} V_{\alpha_\pm}(z, \bar{z}), \quad (80)$$

where α_\pm are the positive and negative solutions of $h(\alpha_\pm) = 1$:

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}. \quad (81)$$

With these ingredients the recipe to compute a CFT correlation function of primary operators $O_{h_i}(z_i, \bar{z}_i)$ using Coulomb gas is (roughly) given by:

$$\begin{aligned} \langle O_{h_1}(z_1, \bar{z}_1) O_{h_2}(z_2, \bar{z}_2) \dots O_{h_N}(z_N, \bar{z}_N) \rangle_{\text{CFT}} = \\ \langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) \dots V_{\alpha_N}(z_N, \bar{z}_N) Q_+^{m_+} Q_-^{m_-} \rangle_{\text{GFF}}. \end{aligned} \quad (82)$$

The subscript GFF indicates that the correlation function is computed with a weight e^{-S} with action (76), while the CFT subscript indicates a correlation function in a conformal field theory. The charges α_i are chosen such that $h(\alpha_i) = h_i$. The number of insertions, m_\pm , of the screening operators is arbitrary, but a constraint which reads

$$m_+\alpha_+ + m_-\alpha_- + \sum_i \alpha_i = 2\alpha_0, \quad (83)$$

must be satisfied. This recipe can be motivated in different ways, but as far as we know cannot be justified fully and rigorously.

A special place is given to operators whose charges $\alpha_{m,n}$ are given by

$$\alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-. \quad (84)$$

The primary operators corresponding to these charges are denoted by $\psi_{r,s}$. r and s are the numbers of screening operator insertions Q_+ and Q_- necessary to compute two-point correlation functions of $\psi_{r,s}$. The operators, $\psi_{r,s}$ have a special role in the representation theory of conformal symmetry (the Virasoro algebra), particularly when r and s are positive integers. In this case correlation functions with insertions of $\psi_{r,s}$ satisfy differential equations of order rs . The relation (84) is called the ‘Kac table’, originally meant to be used only when r and s are positive integers, but often is extended to include all integer and even half integer indices. We shall often make use of this (extended) Kac table parametrization.

We may use (84) as a convenient parametrization of the primary operators appearing in the theory. Every primary operator has a conformal weight h . With this conformal weight, a conformal charge α may be associated, which satisfies $h(\alpha) = h$. In fact, for a given value of α_0 , the Eq. (79) for a conformal weight has two solutions for the Coulomb charge:

$$\alpha = \alpha_0 \pm \sqrt{\alpha_0^2 + h}. \quad (85)$$

These charges can always be written as $\alpha = \alpha_{r,s}$ for some choice of numbers r and s , not necessarily positive integers.

The parametrization (84) has several advantages. First, the conformal charge α naturally appears in formulas, and turns out to be a particularly convenient parametrization. Second, the parametrization is widely used in CFT literature. And third, if α happens to be given by $\alpha_{r,s}$ for some positive integers r and s , there is some chance that the operator will have special properties, closely related to the representation theory of CFT, so keeping track of r and s is generally a good idea.

Notice that given a central charge c , Eq. (77) has two solutions for the possible background charge: $\alpha_0 = \pm\sqrt{(1-c)/24}$. In the SLE language these correspond to two dual values of κ describing distinct ‘phases’ of SLE curves: ‘dilute’ ($\kappa < 4$) and ‘dense’ ($\kappa > 4$).⁷³ A convenient choice of parametrization⁴² is such where

$$2\alpha_0(\kappa) = \frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}, \quad (86)$$

$$\alpha_+(\kappa) = \frac{\sqrt{\kappa}}{2}, \quad \alpha_-(\kappa) = -\frac{2}{\sqrt{\kappa}}. \quad (87)$$

This value of α_0 is positive in the dense phase and negative in the dilute phase. In terms of κ the general Kac table charges and weights become

$$\alpha_{r,s}(\kappa) = \frac{4s - \kappa r + \kappa - 4}{4\sqrt{\kappa}}, \quad (88)$$

$$h_{r,s}(\kappa) = \frac{(\kappa r - 4s)^2 - (\kappa - 4)^2}{16\kappa}. \quad (89)$$

The models we are interested in are related to conformal restriction, and the central charge for them is $c = 0$.

The two values of κ that correspond to it are $\kappa = 8/3$, related to self-avoiding walks, and $\kappa = 6$ related to percolation. Since the boundaries of restriction measures are always simple (“dilute”) curves $\text{SLE}(8/3, \rho)$, in order to describe them we choose $\kappa = 8/3$ and

$$\alpha_0\left(\frac{8}{3}\right) = -\frac{1}{2\sqrt{6}}, \quad \alpha_+\left(\frac{8}{3}\right) = \sqrt{\frac{2}{3}}, \quad \alpha_-\left(\frac{8}{3}\right) = -\sqrt{\frac{3}{2}}. \quad (90)$$

With this choice the charges and weights of operators appearing in the Kac table are

$$\alpha_{r,s}\left(\frac{8}{3}\right) = \frac{3s - 2r - 1}{2\sqrt{6}}, \quad (91)$$

$$h_{r,s}\left(\frac{8}{3}\right) = \frac{(2r - 3s)^2 - 1}{24}. \quad (92)$$

On the other hand, to describe percolation hulls that in the continuum are SLE_6 curves, we need to choose $\kappa = 6$ and

$$\alpha_0(6) = \frac{1}{2\sqrt{6}}, \quad \alpha_+(6) = \sqrt{\frac{3}{2}}, \quad \alpha_-(6) = -\sqrt{\frac{2}{3}}. \quad (93)$$

In this case the general Kac charges and weights are

$$\alpha_{r,s}(6) = \frac{2s - 3r + 1}{2\sqrt{6}}, \quad (94)$$

$$h_{r,s}(6) = \frac{(3r - 2s)^2 - 1}{24}. \quad (95)$$

The conformal covariance properties of the partition functions given in Eqs. (49) and (51) imply that these are CFT correlation functions, where the points p_i and x_i (see Fig. (11)) correspond to the insertion of primary operators of conformal weights h_κ for the SLE traces beginning at x_i and h_i for the points p_i . Moreover, the martingale conditions in the form of the second order differential equations (75) imply that the operators inserted at points x_i are degenerate at level two, to use the CFT language. Using the Kac table parametrization, these can be identified with $\psi_{1,2}$, as was first demonstrated by Bauer and Bernard⁷⁴ for an arbitrary value of κ . Here we provide an interpretation of this operator in the CFT/Coulomb gas language for $c = 0$. In this case the operator has conformal weight $5/8$ and the corresponding Coulomb charge $\alpha = \sqrt{3/8}$ can be written as $\alpha_{1,2}(8/3)$. Thus, the operator creating an $\text{SLE}_{8/3}$ trace is $\psi_{1,2}$.

The CFT interpretation of $\text{SLE}(\kappa, \rho)$ which is anchored at a point x on the boundary, is the insertion of the operator $\psi_{1,2}$ at x and another operator $O_{h'}$ of conformal dimension h' at $x - 0^+$. The relation between h' and ρ is given by Eq. (61). Comparing this equation with (79)

for $\alpha_0(8/3) = -1/(2\sqrt{6})$ ($c = 0$) we see that

$$\alpha' = \sqrt{\frac{3}{32}} \rho \quad (96)$$

is a Coulomb charge representing $O_{h'}$. We have two operators close together at x , $\psi_{1,2}$ and $O_{h'}$. These in fact fuse together to produce another primary operator $O_h(x)$, whose conformal weight depends on ρ as in Eq. (2). A conformal charge consistent with h appearing in (2) is given by

$$\alpha = \alpha_{1,2}(8/3) + \alpha'. \quad (97)$$

This last formula can be read as follows: an operator of charge $\alpha_{1,2}(8/3)$ (which creates the SLE trace) is fused with an operator of charge α' (this is the operator $O_{h'}$) to produce an operator O_h whose charge is given by the simple sum of the two operators which were fused to produce it. This is in fact no surprise, as charges add up in the Gaussian free field formulation of CFT. O_h may then be called the *simple* fusion product of $O_{h'}$ and $\psi_{1,2}$.

We can also consider fusions of multiple $\psi_{1,2}$. These may lead to SLE lines that form small loops touching the boundary, but the simple fusion where the charges add leads to multiple SLE lines starting at the same point and going to infinity of other distant points.^{40,42,45} These are created by insertions of the so-called n -leg operators $\psi_{1,n+1}$. For $\kappa = 8/3$ these operators create n SAWs and have dimensions

$$h_{1,n+1}\left(\frac{8}{3}\right) = \frac{n(3n+2)}{8}. \quad (98)$$

For $\kappa = 6$ they create n percolation hulls, and their dimensions are

$$h_{1,n+1}(6) = \frac{n(n-1)}{6}. \quad (99)$$

Multi-leg operators can also be defined in the bulk. It is known^{40,42} that the bulk n -leg operator is $\psi_{0,n/2}$ in the extended Kac table. The conformal weights of these operators for the two values of κ relevant for $c = 0$ are

$$h_{0,n/2}\left(\frac{8}{3}\right) = \frac{9n^2 - 4}{96}, \quad h_{0,n/2}(6) = \frac{n^2 - 1}{24}. \quad (100)$$

VI. POINT CONTACT CONDUCTANCES IN THE SIMPLEST SETTINGS

In this section we will use the relation (3) between PCCs at critical points of the disordered systems considered above and conformal restriction measures to compute PCCs in the simplest settings.

The simplest quantity one can compute is the average two point conductance $\langle g(a, b) \rangle$ between points a and b on the straight absorbing boundary of a critical system occupying the upper half plane. We have already

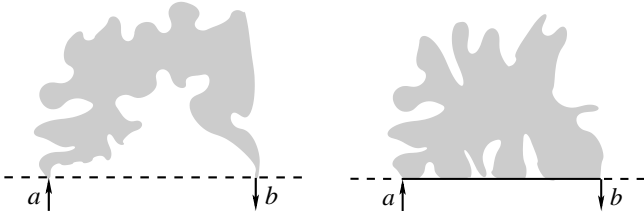


FIG. 12: Two-point point contact conductance. Left: contacts are placed on the absorbing boundary. Right: contacts are placed at juxtapositions of the absorbing and a reflecting boundary.

seen that this conductance is the same as the partition function $Z_{\mathbb{H}}(a, b)$. In CFT language, these are primary (boundary) operators with scaling dimensions h_A . This immediately implies that the same quantity is given by a correlation functions of currents, $\langle g(a, b) \rangle = \langle j_a j_b \rangle$, where the operator j_a injects current into the system through a link at the point a and j_b extracts the current through a link at the point b (see the left panel in Fig. 12).

$$\langle g(a, b) \rangle = \frac{C}{|a - b|^{2h_A}}. \quad (101)$$

We will argue below that for all systems that we consider $h_A = 1$ is the weight of the conserved current operator.

Now let us illustrate how this result can be understood in terms of pictures as samples of a restriction measure, and their SLE boundaries. The pictures that contribute to $g(a, b)$ include two lines, the inner and outer boundaries of the picture, both lines start at a and end at b (see the left panel of Fig. (12)). To define a partition function for the process described by Eqs. (53, 54) that creates those lines, we must split point a and point b each into three points a^- , a^0 and a^+ , b^- , b^0 and b^+ . a^- is now the origin of the outer line, a^+ is the origin of the inner line and at a^0 we place an operator with a proper weight such that after fusing a^- , a^0 and a^+ , we get an operator of a weight corresponding to j . The same is done at b . We end up with a partition function $Z_{\mathbb{H}}(a^0, b^0; a^-, b^-, a^+, b^+)$. The fused partition function, $Z_{\mathbb{H}}(a, b)$, obtained when all the points associated with a are fused and all the points associated with b are fused gives $\langle g(a, b) \rangle = Z_{\mathbb{H}}(a, b)$. The fused partition function also satisfies (51), and these three conditions are sufficient to specify it completely up to a constant factor, and we get back to Eq. (101). The same arguments apply to the average PCC between two contacts placed at juxtapositions of the absorbing boundary and a reflecting boundary (see the right panel in Fig. (12)). Depending on whether the reflecting segment of the boundary (illustrated by the solid line) is “left” or “right” (see Sec. II B for definitions), we get

$$\langle g(a, b) \rangle = \frac{C}{|a - b|^{2h_{LA}}} \quad \text{or} \quad \frac{C}{|a - b|^{2h_{RA}}}. \quad (102)$$

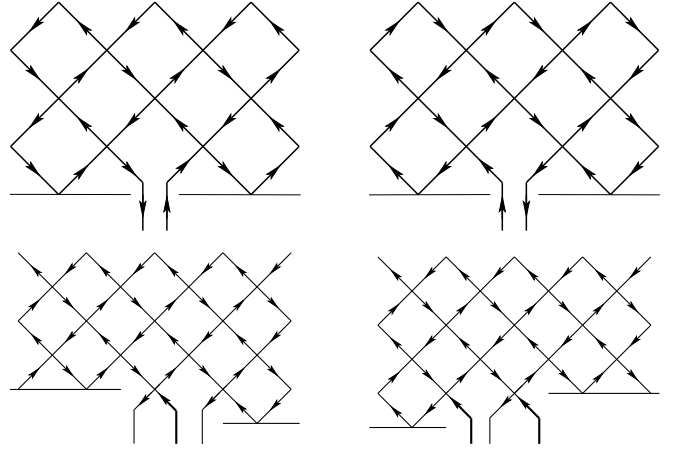


FIG. 13: Top: Point contacts placed at reflecting boundaries through small openings. The dimensions of current insertions at these contacts are denoted by h_R and h_L . Bottom: Point contacts placed at small openings between reflecting boundaries of opposite chirality. The corresponding dimensions are h_{RL} and h_{LR} .

We can also consider point contacts placed at small openings in reflecting boundaries, see the top panel in Fig. 13. We denote the dimensions of current insertions through such contacts by h_R and h_L . The two situations shown on the top in Fig. 13 are related by reflection across the vertical line. Such reflection should not change the dimensions of the current insertions, so we expect these dimensions to be equal:

$$h_R = h_L. \quad (103)$$

Then the two point PCCs between such contacts is

$$\langle g(a, b) \rangle = \frac{C}{|a - b|^{2h_R}}. \quad (104)$$

Point contacts can be also placed at small openings between reflecting boundaries with the opposite chirality, see the bottom part of Fig. 13. Injecting current through the middle (incoming) link on the left figure inserts an operator with dimension h_{RL} . In this situation there is no symmetry relating h_{RL} and h_{LR} , so we expect these dimensions to be different in general.

We can consider slightly more complicated setups without any additional input. Indeed, Eqs. (51) contain three independent conditions which determine all three point functions up to a constant. The three point function of primary operators of dimensions h_a, h_b, h_c inserted at points a, b, c has the form:

$$\frac{C}{|a - b|^{h_a + h_b - h_c} |a - c|^{h_a + h_c - h_b} |b - c|^{h_b + h_c - h_a}}. \quad (105)$$

As a physical example we can consider the following setup shown in the left panel of Fig. 14. It shows a critical system whose boundary is mostly absorbing but

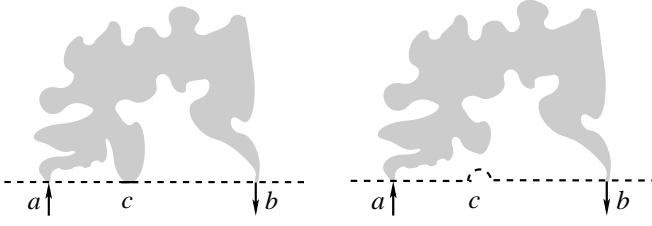


FIG. 14: A three point function. Left: the change $\delta\langle g(a, b) \rangle$ in the two-point conductance is positive when a small segment of reflecting boundary of length ϵ is inserted near the point c . Right: $\delta\langle g(a, b) \rangle$ is negative when a small bump of radius ϵ on the absorbing boundary is inserted near the point c .

with a small insertion (of length ϵ) of a reflecting boundary near the point c . This insertion makes possible for a picture contributing to $\langle g(a, b) \rangle$ to touch this small reflecting boundary segment, thereby increasing the overall conductance. The difference between the conductance in this case and the one without the insertion is then represented by pictures that necessarily touch the reflecting segment. This corresponds to a three point function of two current insertion operators (of dimension h_A) and one operator that “forces” the picture to “touch” the point c . Denoting the dimension of this operator by h_T , we have

$$\delta\langle g(a, b) \rangle = \frac{C\epsilon^{h_T}}{|a-b|^{2h_A-h_T}|a-c|^{h_T}|b-c|^{h_T}}. \quad (106)$$

Similarly, we can consider the situation where the absorbing boundary is slightly moved into the system, forming a small semicircular bump of radius ϵ centered at c . This is shown in the right panel of Fig. 14. In this case the conductance $\langle g(a, b) \rangle$ is reduced by essentially the same amount (106), since now the pictures that were going through the semicircle (the same as the picture on the left panel of Fig. 14) do not contribute anymore.

Another situation, shown in Fig. (15), leads to a three-point function. Here a reflecting boundary extends from a to b , and the rest of the boundary is absorbing. The current is injected into the sample at the point b and extracted through a point c on the absorbing boundary. The weight of the operator at c is h_A . The weight h_b of the operator at b is either h_{LA} or h_{RA} , and depends on the particular situation. Point a is a point at which the boundary conditions change, and a priori we do not know if it may be described by a primary operator. We do know however that for every picture (the gray area on Fig. (15)) there is a point x between a and b at which the picture will lift off the real axis, never to visit the real axis again before exiting at c . For all pictures with a given point x , the boundary to the left of x serves as the absorbing boundary (microscopically, a shift of one lattice spacing into the bulk is necessary). The fact that the boundary conditions to the left of x are effectively absorbing allows us to identify x as a point anchoring a one sided restriction measure. Indeed, the left bound-

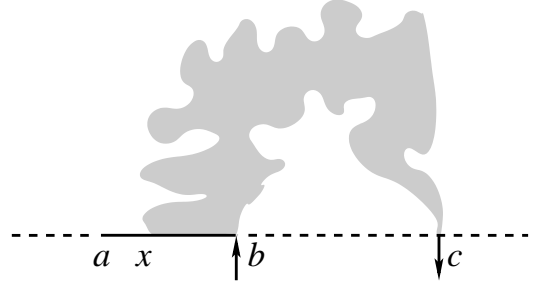


FIG. 15: Feynman picture drawn in gray for current inserted at an interface between reflecting and absorbing boundaries (the reflecting boundary, represented by a heavy line, occupies the segment $[a, b]$, the dotted line represents absorbing boundaries). The point x denotes the location at which the diagram lifts-off the real axis.

ary of the picture is anchored at x and obeys restriction with respects to sets placed on the absorbing boundary to the left of x . Denoting the dimension of the operator at the point x by h_l , the contribution $\langle g(b, c; a, x) \rangle$ of the pictures with a given x to the conductance $\langle g(b, c; a) \rangle$ is given by the three-point function

$$\frac{C}{|x-b|^{h_l+h_b-h_A}|x-c|^{h_l+h_A-h_b}|b-c|^{h_b+h_A-h_l}} \quad (107)$$

Summing (integrating) over all lift-off points x we get

$$\langle g(b, c; a) \rangle = \int_a^b dx \langle g(b, c; a, x) \rangle. \quad (108)$$

While for arbitrary weights h_l, h_A, h_b this integral can be expressed in terms of a hypergeometric function, the actual final expression (see Eq. (114) below) is quite simple due to the special values $h_A = h_l = 1$.

In the next subsections we will discuss the weights h_A, h_T, h_l , and $h_{L,R}$ in the models that we consider in this paper. We will argue that the weights $h_A = 1$, $h_T = 2$, and $h_l = 1$ are *super-universal* and do not depend on a particular model and its symmetry class. We will exhibit other operators with super-universal weights, and will give explicit constructions of these for specific systems of our interest. Other weights ($h_{L,R}$) will also be discussed for each system separately.

A. Super-universal weights

In this subsection we show that the weights of some operators that have appeared in the previous sections are super-universal, that is, they do not depend on the symmetry class of a particular disordered system, but only on the conformal restriction property.

The first example is the operator that injects current through an absorbing boundary. Its dimension $h_A = 1$ is super-universal, which is related to the fact that we are describing a critical conductor with a finite average

conductivity and various conductances. Indeed, suppose two leads are connected to the conductor: one lead connects to a segment $[a, b]$ of the boundary, while the other connects to the segment $[c, d]$. The conductance between the leads is given by the Kubo (Landauer) formula

$$\langle g \rangle = \sum_{i \in [a, b]} \sum_{j \in [c, d]} \langle j_i j_j \rangle, \quad (109)$$

where $i \in [a, b]$, for example, denotes that site i belongs to segment $[a, b]$. In the continuum limit this becomes

$$\langle g \rangle = \epsilon^{2h_A - 2} \int_{[a, b]} \int_{[c, d]} dx dy \langle j(x) j(y) \rangle, \quad (110)$$

where h_A is the conformal weight of j , and ϵ is the lattice spacing. For this expression to remain finite (neither zero, nor infinity) as expected for a critical conductor, the weight h_A must be equal to 1. If $h_A > 1$ we will have an insulator as $\epsilon \rightarrow 0$, while if $h_A < 1$ we will have a superconductor.

Another way to arrive at the same conclusion is to consider the total current through a segment $I = \int_a^b dx j(x)$. This total current should be a conformally-invariant object of dimension 0. But if we perform a conformal transformation that takes x to $f(x)$, the current transforms to

$$I = \int_{f(a)}^{f(b)} df |f'(x)|^{h_A - 1} j(f). \quad (111)$$

The conformal invariance of I implies $h_A = 1$.

The second example of a super-universal operator appears in the situations shown in Fig. 14. In these cases we consider diagrams in which current is forced to pass through a point x on the boundary of the system. The boundary is absorbing around this point. A diagram forced to touch the point x looks locally as a sample of a two-sided restriction measure, and the weight of this measure can be easily seen to be 2. Indeed, to select only diagrams that pass through the point x we introduce a small semicircle A of radius ϵ centered at x . The difference between the overall weight of all diagrams and the weight of diagrams that avoid A is proportional to the weight of the pictures that pass through x (as ϵ tends to zero). The conformal restriction property allows to compute the weight of the diagrams that avoid A by effecting a conformal transformation that removes A , see Eq. (48). Such transformation $f(z)$ that also fixes the points a and b can be easily written explicitly (see Appendix B). Then using Eq. (48) we can find the change in the PCC after the deformation of the boundary to the

first non-vanishing order in ϵ as

$$\begin{aligned} \delta \langle g(a, b) \rangle &= (1 - |f'(a)|^{h_A} |f'(b)|^{h_A}) \langle g(a, b) \rangle \\ &\approx \frac{2h_A |b - a|^2}{|c - a|^2 |c - b|^2} \epsilon^2 \langle g(a, b) \rangle \\ &= \frac{C \epsilon^2}{|a - b|^{2h_A - 2} |a - c|^2 |b - c|^2} \\ &= \frac{C \epsilon^2}{|a - c|^2 |b - c|^2}. \end{aligned} \quad (112)$$

This is exactly the equation (106) with $h_T = 2$, a universal value independent, in particular, of $h_A = 1$.

The dimension $h_T = 2$ is consistent with the fact that in the CFT language the removal of the semicircle A is effected by the insertion of the stress energy tensor T . The stress energy tensor T has always dimension 2, and for $c = 0$ is a primary operator.

The third super-universal weight is that of lift-off points: $h_l = 1$, see Fig. 15. This can be seen in the following way: the lift-off point must always be integrated over to obtain a physically measurable average conductance. The conductance is conformal invariant. For example, to compute the overall contribution of diagrams in Fig. (15) we must take $\langle \int_a^b dx O_{h_l}(x) \dots \rangle$, where $O_{h_l}(x)$ is the operator which creates a lift-off at point x , and the ellipsis denote other operators which must be inserted (in the example shown in Fig. 15 the other operators are inserted at the points b and c). Upon a Möbius conformal transformation $f(z)$ that maps the upper half plane to the upper half plane, the integral $\int_a^b dx O_{h_l}(x)$ transforms to

$$\int_{f(a)}^{f(b)} df |f'(x)|^{h_l - 1} O_{h_l}(f). \quad (113)$$

Conformal invariance of the PCC that is obtained from this expression implies that $h_l = 1$.

Since the operator O_{h_l} is of dimension one, its integral is also a primary operator, but of dimension⁷⁵ $h_{BC} = 0$. This means that in Fig. 15, instead of summing over x we may simply place an operator of dimension zero at a . Thus, the operator that changes the boundary condition from absorbing to reflecting is primary of dimension zero. Note that this result is completely general. This has been noticed and used before by Cardy⁷⁶ in the context of percolation to compute the crossing probability in a rectangle. But for other systems with central charge $c = 0$, to the best of our knowledge, the value $h_{BC} = 0$ represents a new result. This shows that the integral expressing the average conductance in Eq. (108) is simply given by the three point function of operators with weights $h_{BC} = 0$, h_b , and $h_A = 1$:

$$\langle g(b, c; a) \rangle = \frac{C}{|a - b|^{h_b - 1} |b - c|^{h_b + 1} |a - c|^{1 - h_b}}. \quad (114)$$

One more super-universal weight h_0 is a special case of

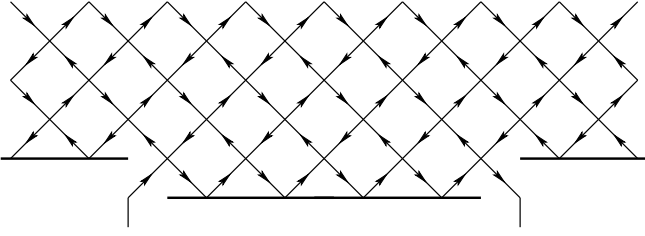


FIG. 16: Contacts placed at points where the two reflecting boundary conditions, left and right, are switched. The average PCC between these contacts does not depend on the distance between them due to the current conservation.

h_{RL} (or h_{LR}) which is obtained when current is injected through the smallest possible opening between reflecting boundaries with the opposite chirality, see Fig. 16. In this situation the current cannot exit through the same opening, and the average PCC between two such contacts does not depend on the distance between them due to current conservation. Therefore, the dimension of such current insertion is zero: $h_0 = 0$.

B. Weights of operators for the SQHE

We now turn to the Kac table classification of the operators appearing in the SQHE problem. These are directly related to hulls of percolation clusters.¹⁶ The operator creating n percolation hulls (multiple SLE_6) at a boundary (also called the boundary n -leg operator) is known⁷⁷ to be $\psi_{1,n+1}$ in the theory with $\kappa = 6$. According to Eq. (95) its weight is

$$h_{1,n+1}(6) = \frac{n(n-1)}{6}. \quad (115)$$

Consider now the operator that injects current through a single link (a point contact) at the origin on the absorbing boundary. This creates a percolation hull that is not allowed to hit the boundary until it goes out through the other point contact. Consider the positive real axis. By known⁷⁸ percolation arguments this absorbing boundary can be replaced by a reflecting boundary plus insertion of an additional hull emanating from a point immediately to the right of the origin. This extra hull “screens” the current carrying hull from approaching the boundary. The same can be done for the negative real axis. This absorbing boundary can also be equivalently described by a reflecting boundary plus insertion of a hull, this time immediately to the left of the origin. The overall outcome is that we can think of current inserted on the absorbing boundary as 3 hulls inserted on a reflecting boundary. This is the three leg operator $\psi_{1,4}$ with dimension $h_A = h_{1,4}(6) = 1$, the super-universal value. This value is consistent with what is known rigorously⁴³ about the SLE_6 conditioned not to touch the boundary.

Similarly, we can consider the operators that inject cur-

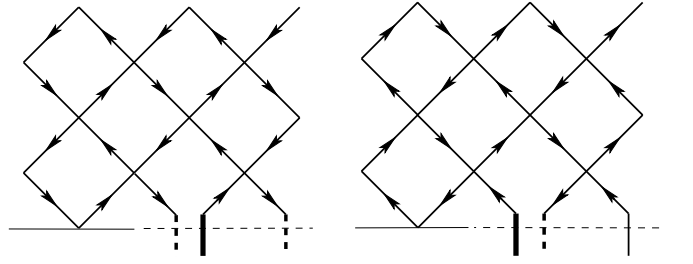


FIG. 17: Point contacts at the juxtapositions of the absorbing boundary with the right and left boundaries. The current is injected through the fat links. The dashed links represent the origins of “screening” percolation hulls, see the main text.

rent at points of juxtaposition of a reflecting and the absorbing boundaries. These are shown in Fig. 17 and have weights h_{RA} and h_{LA} . Let the current be injected at the origin through links that are shown by fat lines. The absorbing boundary along the the positive real axis can be replaced by a reflecting boundary plus a “screening” hull inserted immediately to the right of the origin (its origin is marked by a dashed link immediately to the right of the current insertion). The reflecting boundary conditions on the negative real axis are in fact different for h_{RA} and h_{LA} . For h_{LA} the negative real axis is simply the reflecting boundary in the percolation language. Then it is clear that the weight h_{LA} is that of the two-leg operator in percolation, that is, $h_{LA} = h_{1,3}(6) = 1/3$. At the same time, for h_{RA} the negative real axis is described by a reflecting boundary plus an insertion of an additional screening hull immediately to the left of the origin. This makes h_{RA} the weight of the three-leg operator just like h_A : $h_{RA} = h_{1,4}(6) = 1$.

The dimensions for operator insertions at reflecting boundaries are easily seen to correspond to the two-hull operators: $h_R = h_L = h_{1,3}(6) = 1/3$. They are equal to each other, in accord with Eq. (103). Similarly, it is easy to see that $h_{RL} = h_{1,4}(6) = 1$ and $h_{LR} = h_{1,3}(6) = 1/3$.

Other super-universal weights are also easily confirmed for the SHQ transition. Indeed, forcing a hull to go through a point on the absorbing boundary is equivalent to have two percolation hulls starting at this point, plus two screening hulls to screen the absorbing boundary. Thus, the relevant operator is the four-leg operator with $h_T = h_{1,5}(6) = 2$. For a lift-off point we have the two halves of the current-carrying hull that comes to this point and leaves it, plus one screening hull that ensures that the current does not touch the boundary again. This is the three-leg operator, and $h_l = h_{1,4}(6) = 1$. The switch between the absorbing and a reflecting boundary can be described by a single screening hull, and the corresponding weight is $h_{BC} = h_{1,2}(6) = 0$. The same one-leg operator injects current through a switch between the left and the right reflecting boundaries (see Fig. 16), and $h_0 = h_{1,2}(6) = 0$.

We comment here that the degeneracies between certain weights ($h_A = h_{RA} = h_{RL}$ and $h_{LA} = h_R = h_L =$

	IQH transition	SQH transition	Classical CC model
h_A	1	$h_{1,4}(6) = 1$	1
h_T	2	$h_{1,5}(6) = 2$	2
h_l	1	$h_{1,4}(6) = 1$	1
h_{BC}	0	$h_{1,2}(6) = 0$	0
h_{RA}	0.8	$h_{1,4}(6) = 1$	$1/2 + \theta_R^H/\pi$
h_{LA}	0.32	$h_{1,3}(6) = 1/3$	$1/2 + \theta_L^H/\pi$
h_R	—	$h_{1,3}(6) = 1/3$	—
h_L	—	$h_{1,3}(6) = 1/3$	—
h_{RL}	—	$h_{1,4}(6) = 1$	—
h_{LR}	—	$h_{1,3}(6) = 1/3$	—
h_0	0	$h_{1,2}(6) = 0$	0

TABLE I: Restriction exponents. The first four lines represent the super-universal weights. The values of h_{RA} and h_{LA} for the IQH are obtained from numerical simulations of PCC in the CC model in Ref. [61]. All other dimensions are exact. The angles $\theta_{R,L}^H$ are the Hall angles at the two types of reflecting boundaries. The symbols “—” mean that the corresponding exponents are not known to us.

h_{LR}) are a consequence of the *locality* property of percolation: even one “absorbing” link next to a point contact creates a screening hull.

C. Weights of operators for classical CC

The super-universal weights are also easily confirmed for the classical CC model (diffusion in a magnetic field). First consider injecting current through the absorbing boundary. The dimension for this is $h_A = 1$. Indeed, since there are no reflecting boundaries around, the problem is equivalent to that of diffusion. It is easy to see, using Green’s functions, that the fraction of diffusing particles that will reach a height y from the real axis if they are released a from the point $i\epsilon$ is ϵ/y , the rest will be trapped by the absorbing boundary. Since releasing the diffusing particles is equivalent to the insertion of current through the absorbing boundary (we assume that ϵ is the lower cutoff scale), and the ϵ dependence marks the weight of the operator, we indeed see that the weight for current insertion through the absorbing boundary is $h_A = 1$. The rigorous formulation of this result in terms of Brownian excursions and its proof are due to Virag.⁷⁹

The weights for current insertions at juxtapositions of the absorbing and reflecting boundaries have been discussed in the sec. III C, with the result $h_{L,R} = 1 - \theta_{L,R}/\pi$. Now let us consider lift-off points. A touch-off point is a point at which current arrives at a point on a reflecting boundary, immediately leaves it, and is conditioned never to touch the boundary again, say, to the left of the point at which it arrived. In the classical CC, due to lack of interference, one may separate the past of the diffusing particle from its future, relative to the moment it reached

the lift-off point. The past and the future are actually two independent restriction measures, whose weights simply add up⁴³ due to lack of interference. The weight of the future measure is $1 - \theta/\pi$ as in Eq. (36). For the past measure we must apply time reversal in order to be able to apply Eq. (36), this takes $\theta \rightarrow \pi - \theta$, which means that for this measure, $h = \theta/\pi$. The sum of the two weights is the weight of the touch of point $h_l = 1$.

The simple decoupling (independence) of the past from the future also occurs if we force the current to pass through a boundary point. Here both the past and the future are current insertion (extraction) operators having the same weight $h_A = 1$. Summing up the weights of the two measures we obtain that the operator in question has weight $h_T = 2$.

We summarize the weights of some operators for the systems of our interest in table I.

VII. CONCLUSIONS AND OUTLOOK

We have revisited the problem of the plateau transition in the integer quantum Hall (IQH) effect and related Anderson localization-delocalization transitions in two spatial dimensions. Specifically, we have considered the Chalker-Coddington network model and related models. In all cases we have focused on the so-called boundary point-contact conductances (PCCs) at critical points, and their behavior in the presence of various boundaries (absorbing and reflecting). While most of our results are general and apply to all problems we consider, let us concentrate here on the most interesting case, the IQH problem.

There are two key observations that allow us to analyze the problem. The first observation is that microscopic expressions for PCCs can be written as a sum of *positive* contributions related to certain geometric objects that we call *pictures*, see Section III A. Written as a sum over pictures, a PCC can be interpreted as a partition function of an ensemble of pictures, each picture having a certain statistical weight. The second observation is that these statistical weights are *intrinsic* and satisfy the so-called restriction property with respect to *absorbing* boundaries. Namely, whenever we deform an absorbing boundary, the pictures that continue to contribute to a PCC are the ones that are present in the new (deformed) system. The pictures that intersect the deformed boundary do not contribute any more, and the PCC is renormalized. When we combine the restriction property with the assumed conformal invariance at the IQH transition point, we can employ the recently developed mathematical theory of conformal restriction measures. This theory is closely related to conformal field theories (CFTs) with zero central charge. As a result, we get several results that were already mentioned in the introduction and discussed at length in the main part of the paper. Let us briefly repeat them here.

First, PCCs in various geometries can be studied as

correlation functions of (Virasoro) *primary* CFT operators. This statement alone allows to calculate exact forms for PCCs that reduce to three-point functions. Secondly, we predict the values of conformal dimensions of some of the primary operators that appear in the theory based on very general arguments. Finally, the connection with conformal restriction is established for other disordered systems, including the spin quantum Hall transition, the classical limit of the Chalker-Coddington model (diffusion in a magnetic field), and the metal in class D. For these systems many more dimensions of primary operators can be obtained exactly.

The relation between the Chalker-Coddington model and conformal restriction that we have discovered allows for a new approach to the study of the critical properties of the IQH transition. A full understanding of the transition by these methods will require much more work. We plan to extend the present paper in several directions.

We hope to be able to consider PCCs in more complicated geometries, where the necessary CFT correlation functions will be four-point or higher functions. In these cases knowing the conformal dimensions of the primary operators involved will not be sufficient. In addition we would have to understand whether there are degenerate operators²⁸ related to existence of null vectors in the corresponding representations of the Virasoro algebra. Potentially, this can be established by analyzing fusions of the operators that we have already identified.

In this paper we have considered only boundary PCCs. The necessary mathematical theory of sets K that “touch” the boundary of a domain at two points is called the “chordal restriction theory”. In principle one can define conductances between point contacts in the bulk of a network (obtained by cutting some of the internal links),⁵¹ or between a boundary and a bulk contact. Upon disorder averaging these conductances should also satisfy a (suitably modified) restriction property. The corresponding mathematical theory of the “bulk” restriction or the “radial” restriction has not been worked out, and we plan to develop it, and its relations to bulk CFT operators.

For the bulk theory the cut points that we have mentioned in Sections II A and III A will likely be important objects. It is known from the conformal restriction theory⁴⁴ that two-sided restriction measures have cut points for any restriction exponent h in the range $5/8 \leq h < 35/24$. The cut points form a fractal set of Hausdorff dimension

$$\begin{aligned} d_{\text{cut}}(h) &= 2 - \frac{(\sqrt{24h+1} - 1)^2 - 1}{12} \\ &= 2 - 2h + \frac{2\sqrt{24h+1} - 1}{12}. \end{aligned} \quad (116)$$

As we have argued, in our problems two-sided restriction measures correspond to PCCs between contacts placed on the absorbing boundary, in which case the restriction exponents is the dimension of the conserved current op-

erator: $h_A = 1$. Then the dimension of the set of cut points is $d_{\text{cut}}(1) = 3/4$. In the case of the SQH transition (percolation), this dimension can be written as

$$d_{\text{cut}}(1) = 2 - 2h_{0,2}(6), \quad (117)$$

where $h_{0,2}(6) = 5/8$ is the dimension of the *bulk* four leg operator (see Eq. (100) for $n = 4$) well known to be related to the critical exponent of the correlation length for percolation

$$\nu_{\text{perc}} = (2 - 2h_{0,2}(6))^{-1} = d_{\text{cut}}(1)^{-1} = 4/3. \quad (118)$$

While we know that the localization length exponent for the SQH transition is exactly this $\nu_{\text{SQH}} = 4/3$, it is very different for the IQH transition. The relation of cut points to ν_{IQH} (if any) is not clear to us at the moment, but it is tantalizing to speculate that one can get more understanding by focusing on a decomposition of fillings of pictures into irreducible components.

Another possible extension of our results is in the direction of studying the restriction property away from critical points. While the conformal invariance is lost away from critical points, the restriction property for pictures survives, and one can attempt to create a theory of “massive” restriction measures. Similar attempts to develop a theory of “off-critical” or “massive” variants of SLE exist in the literature.^{80,81} Also, one-sided conformal restriction measures can be built from Brownian motions with oblique reflection at boundaries, and one can try to extend this construction by introducing a finite “killing rate” for the Brownian particles. In the field theory language this corresponds to adding a mass term to the action (39).

Finally, we hope that the general idea of using notions and methods of stochastic conformal geometry (conformal restriction and SLE) can be fruitful in the study of other critical disordered systems.

VIII. ACKNOWLEDGEMENTS

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Appendix A: Directed graphs, pictures, and Feynman paths

In this Appendix we present a relation between pictures and Feynman paths on the CC network. The con-

nection goes through the notion of a directed graph, or *digraph*, see Refs. [82–85]. Namely, for each picture p we construct a digraph with vertices being the network nodes visited by the picture by replacing each link of the picture that is traversed n_j times with n_j directed edges. (All graphs constructed in this way are *loopless*: there is no edge in them that connects a vertex to itself.) Then each Feynman path $f \in F(p)$ corresponds to an Eulerian trail (that is, a sequence of directed edges that visits every edge exactly once) on this digraph. The correspondence is one-to-many, since all permutations among n_j edges of the digraph connecting a pair of vertices correspond to the same Feynman path.

There are several theorems in the theory of directed graphs that are relevant for our discussion and allow us to characterize the pictures that come from Feynman paths, and also count the number of Feynman paths $|F(p)|$ for a given picture p .

First, we introduce a few definitions. Let D be a digraph with vertex set $V = \{v_1, \dots, v_m\}$ and edge set $E = \{e_1, \dots, e_n\}$. A *trail* in D is a sequence e_1, e_2, \dots, e_r of distinct edges such that the final vertex of e_i is the initial vertex of e_{i+1} for all $1 < i < r - 1$. If, in addition, the final vertex of e_r is the initial vertex of e_1 , then the trail is called a *tour* or *cycle*. A trail (tour) is *Eulerian* if it visits every edge of D exactly once. A digraph that has no isolated vertices and contains an Eulerian tour is called an Eulerian digraph. The *outdegree* of a vertex v , denoted $\text{outdeg}(v)$, is the number of edges of the digraph with initial vertex v . Similarly the *indegree* of v , denoted $\text{indeg}(v)$, is the number of edges of the digraph with final vertex v . A digraph is *balanced* if $\text{indeg}(v) = \text{outdeg}(v)$ for all vertices v .

The first theorem that we need is the following: a digraph without isolated vertices is Eulerian if and only if it is connected and balanced. This immediately gives the characterization of pictures that come from Feynman paths: for every vertex of such a picture the sum of numbers n_j on the incoming links must be equal to the sum of numbers n_j on the outgoing links. An example of such a balanced picture is shown in Fig. 18. Actually, there is a little caveat that we need to mention. As drawn, this picture has the initial and final vertices that are *not* balanced. To eliminate this problem, we connect these two vertices into a single vertex v_0 (labeled by 0 in Fig. 18) with $\text{outdeg}(v_0) = \text{indeg}(v_0) = 1$. The resulting picture *is* balanced. Moreover, after this is done, the number of Feynman paths corresponding to the original picture is equal to the number of Eulerian tours on the digraph corresponding to the modified picture divided by the multiplicity factors $n_j!$ for each link of the balanced picture.

Next we describe how we can count the number of Feynman paths $|F(p)|$ that correspond to a balanced picture p . We need two more theorems. One of them is the so-called BEST theorem that relates the number of Eulerian tours on a digraph D to the number of spanning (directed) trees on D . Here is a precise formulation. Let

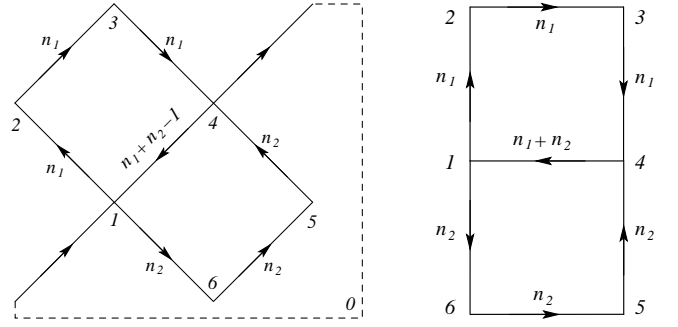


FIG. 18: Left: a balanced picture (see the main text). The dashed line joins the initial and final vertices of every Feynman path corresponding to this picture into a single vertex labeled by 0 here. Removing this vertex leads to the balanced picture shown on the right.

D be a connected balanced digraph with vertex set V . Fix an edge e of D , and let v be the initial vertex of e . Let $T(D, v)$ denote the number of oriented (spanning) subtrees of D with root v , and let $E(D, e)$ denote the number of Eulerian tours of D starting with the edge e . Then

$$E(D, e) = T(D, v) \prod_{u \in V} (\text{outdeg}(u) - 1)!. \quad (\text{A1})$$

The other theorem, the so-called matrix-tree theorem that gives the number of spanning trees $T(D, v)$ with a given initial vertex v in terms of the minor of the *Laplacian matrix* of the digraph. Let us denote the number of edges going from vertex v_i to vertex v_j by m_{ij} . The Laplacian matrix $L = L(D)$ of a directed graph D with vertex set $V = \{v_1, \dots, v_m\}$ is the $m \times m$ matrix

$$L_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j, \\ \text{outdeg}(v_i) & \text{if } i = j. \end{cases} \quad (\text{A2})$$

The matrix-tree theorem states: Let D be a digraph with vertex set $V = \{v_1, \dots, v_m\}$, and let $1 \leq k \leq m$. Let L be the Laplacian matrix of D , and define L_k to be L with the k -th row and column deleted. Then

$$T(D, v) = \det L_k. \quad (\text{A3})$$

The result is independent of k .

Combining Eqs. (A1, A3) with the known degeneracy of the Eulerian tours that give the same Feynman path, we finally obtain the following general formula:

$$|F(p)| = \det L_k \frac{\prod_{u \in V} (\text{outdeg}(u) - 1)!}{\prod_{i,j} m_{ij}!}. \quad (\text{A4})$$

To illustrate this formula, consider the balanced picture shown on the left in Fig. 18. Instead of counting of the number of Eulerian trails that start at the beginning of every Feynman path, we can join the initial and fi-

nal vertices of these paths into a single vertex (labeled by 0 on the left in the figure), and count the number of Eulerian tours on the corresponding digraph. It is clear from the above discussion that the extra vertex 0 does not enter into the calculation of $E(D, e)$ (even though the denominator in the formula (A4) for $|F(p)|$ should still contain the edge multiplicities m_{ij} from the original picture). Therefore, we remove it and obtain the balanced picture shown on the right in Fig. 18. Labeling the remaining vertices as shown, we obtain the following Laplacian matrix:

$$L = \begin{pmatrix} n_1 + n_2 & -n_1 & 0 & 0 & 0 & -n_2 \\ 0 & n_1 & -n_1 & 0 & 0 & 0 \\ 0 & 0 & n_1 & -n_1 & 0 & 0 \\ -n_1 - n_2 & 0 & 0 & n_1 + n_2 & 0 & 0 \\ 0 & 0 & 0 & -n_2 & n_2 & 0 \\ 0 & 0 & 0 & 0 & -n_2 & n_2 \end{pmatrix}. \quad (\text{A5})$$

Deleting the first row and the first column, we get

$$\det L_1 = (n_1 + n_2)n_1^2n_2^2. \quad (\text{A6})$$

The formula (A4) now gives

$$\begin{aligned} |F(p)| &= (n_1 + n_2)n_1^2n_2^2 \\ &\times \frac{[(n_1 + n_2 - 1)!(n_1 - 1)!(n_2 - 1)!]^2}{(n_1 + n_2 - 1)![n_1!n_2!]^3} \\ &= \frac{(n_1 + n_2)!}{n_1!n_2!}. \end{aligned} \quad (\text{A7})$$

In this particular case $|F(p)|$ has a combinatorial interpretation as the number of distinct orderings of going around the top and the bottom plaquettes on the right picture in Fig. 18. However, in the more complicated cases there is no such simple interpretation, while the general formula (A4) is still straightforward to use. For example, for the picture shown in Fig. 19 we have

$$|F(p)| = \frac{(n_1 + n_2 + n_3 - 1)!}{n_1!(n_2 - 1)!n_3!} + \frac{(n_1 + n_2 + n_3 - 2)!}{(n_1 - 1)!n_2!(n_3 - 1)!}. \quad (\text{A8})$$

We note here that the weighting factors for Feynman paths f that enter the definition of the quantity $S(p)$ in Eqs. (11–13) are not determined by the graph-theoretic data for the corresponding digraph (even in the simplest case of the critical point in the isotropic system). Therefore, it seems that an explicit calculation of $S(p)$ for a given picture p is a much more challenging problem than that for $|F(p)|$.

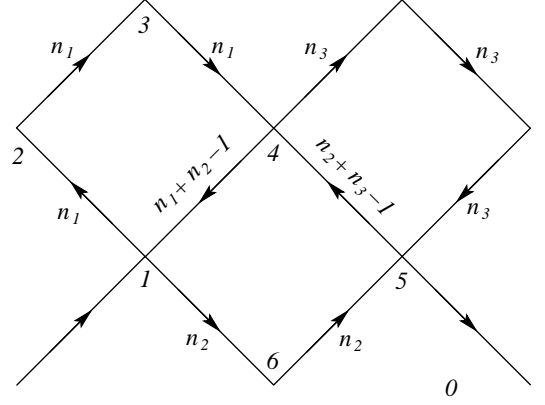


FIG. 19: A more complicated balanced picture.

Appendix B: A conformal map

Here we construct the conformal map $f : \mathbb{H} \setminus A \rightarrow \mathbb{H}$, where A is the semicircle of radius ϵ centered at $c \in \mathbb{R}$, that preserves two points $a < c$ and $b > c$, see Fig. 14. The semicircle's diameter along the real axis goes from $c_- \equiv c - \epsilon$ to $c_+ \equiv c + \epsilon$. We are interested in the limit where ϵ is much smaller than other distances in the problem: $\epsilon \ll c - a, b - c$, and will expand results to the first non-trivial order in ϵ .

Let the original domain $\mathbb{H} \setminus A$ be in the complex z plane. We construct the map $w(z)$ in stages. First, we perform a Möbius transformation $s(z)$ that maps the point a to 0 and the point b to ∞ :

$$s(z) = \frac{z - a}{b - z}. \quad (\text{B1})$$

The images of various points under this map are

$$\begin{aligned} s(a) &= 0, \quad s(b) = \infty, \\ s(c_{\pm}) &= \frac{c_{\pm} - a}{b - c_{\pm}} \approx \frac{c - a}{b - c} \left[1 \pm \epsilon \left(\frac{1}{c - a} - \frac{1}{b - c} \right) \right]. \end{aligned} \quad (\text{B2})$$

Since $s(z)$ is a Möbius transformation, the semicircle A maps to another semicircle in the s plane. The center of this semicircle is

$$s_0 = \frac{1}{2} [s(c_+) + s(c_-)] = \frac{(c - a)(b - c) + \epsilon^2}{(b - c)^2 - \epsilon^2} \approx \frac{c - a}{b - c}, \quad (\text{B3})$$

and its radius is

$$r_0 = \frac{1}{2} [s(c_+) - s(c_-)] = \frac{\epsilon(b - a)}{(b - c)^2 - \epsilon^2} \approx \epsilon \frac{b - a}{(b - c)^2}. \quad (\text{B4})$$

Next we shift everything by s_0 and rescale by r_0 :

$$t(s) = \frac{s - s_0}{r_0} = \frac{z(1 + s_0) - a - bs_0}{r_0(b - z)}. \quad (\text{B5})$$

This transformation preserves the infinity, but maps 0 to

$$t_0 = -\frac{s_0}{r_0} = -\frac{(c-a)(b-c) + \epsilon^2}{\epsilon(b-a)} \approx -\frac{1}{\epsilon} \frac{(c-a)(b-c)}{(b-a)}, \quad (\text{B6})$$

and the semicircle in the s plane to the semicircle of unit radius centered at the origin in the t plane.

Now we can perform the Zhukovsky transformation

$$\begin{aligned} u(t) &= \frac{1}{2} \left(t + \frac{1}{t} \right) \\ &= \frac{1}{2} \left(\frac{z(1 + s_0) - a - bs_0}{r_0(b - z)} + \frac{r_0(b - z)}{z(1 + s_0) - a - bs_0} \right), \end{aligned} \quad (\text{B7})$$

which removes the semicircle in the t plane, preserves the infinity, and maps t_0 to

$$u_0 = \frac{1}{2} \left(t_0 + \frac{1}{t_0} \right) \approx \frac{t_0}{2} = -\frac{1}{2\epsilon} \frac{(c-a)(b-c)}{(b-a)}. \quad (\text{B8})$$

One more Möbius transformation maps ∞ back to b , and u_0 to a :

$$w(u) = \frac{b(u - u_0) + a}{u - u_0 + 1}. \quad (\text{B9})$$

Finally, the transformation $w(z)$ that we want is obtained by composing all the above maps:

$$f = w \circ u \circ t \circ s. \quad (\text{B10})$$

Under these maps the points a and b are successively mapped as

$$a \xrightarrow{s} 0 \xrightarrow{t} t_0 \xrightarrow{u} u_0 \xrightarrow{w} a, \quad b \xrightarrow{s} \infty \xrightarrow{t} \infty \xrightarrow{u} \infty \xrightarrow{w} b.$$

Now we can wind the derivative of the map $f(z)$ as

$$\begin{aligned} f'(z) &= w'(u) \cdot u'(t) \cdot t'(s) \cdot s'(z) \\ &= \frac{1}{2r_0} \left(1 - \frac{1}{t_0^2} \right) \left[\frac{b-a}{(b-z)(u-u_0+1)} \right]^2. \end{aligned} \quad (\text{B11})$$

From this expression we immediately see that

$$f'(a) = \frac{1}{2r_0} \left(1 - \frac{1}{t_0^2} \right). \quad (\text{B12})$$

To evaluate $f'(b)$ we need first to find

$$\begin{aligned} (b-z)u &= \frac{z(1+s_0) - a - bs_0}{2r_0} + \frac{r_0(b-z)^2}{z(1+s_0) - a - bs_0}, \\ (b-z)u|_{z=b} &= \frac{b-a}{2r_0}. \end{aligned} \quad (\text{B13})$$

Then

$$f'(b) = 2r_0 \left(1 - \frac{1}{t_0^2} \right). \quad (\text{B14})$$

The basic transformation formula for conformal restriction measures, Eq. (48), now gives

$$\begin{aligned} Z_{\mathbb{H} \setminus A}(a, b) &= |f'(a)|^{h_A} |f'(b)|^{h_A} Z_{\mathbb{H}}(a, b) \\ &= \left(1 - \frac{1}{t_0^2} \right)^{2h_A} Z_{\mathbb{H}}(a, b). \end{aligned} \quad (\text{B15})$$

Relating this to transport properties, we find the change in the average point contact conductance between a and b upon deforming the real axis by the bump A :

$$\begin{aligned} \delta \langle g(a, b) \rangle &= (1 - |f'(a)|^{h_A} |f'(b)|^{h_A}) \langle g(a, b) \rangle \\ &= \left[1 - \left(1 - \frac{1}{t_0^2} \right)^{2h_A} \right] \langle g(a, b) \rangle \\ &\approx \frac{2h_A}{t_0^2} \langle g(a, b) \rangle \approx \frac{2h_A |b-a|^2}{|c-a|^2 |c-b|^2} \epsilon^2 \langle g(a, b) \rangle \\ &= \frac{C \epsilon^2}{|a-b|^{2h_A-2} |a-c|^2 |b-c|^2}. \end{aligned} \quad (\text{B16})$$

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